THE SIXTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE sixteenth annual meeting of the Society was held at Boston on Tuesday, Wednesday, and Thursday, December 28–30, 1909, in affiliation with the American association for the advancement of science. Tuesday afternoon was devoted to a joint session with Sections A and B of the Association. A joint session was held with section A on Wednesday morning, the programme consisting of Professor Keyser's vice-presidential address "On the thesis of modern logistic," a report by Professor D. E. Smith on "The work of the International Commission on the teaching of mathematics," and the papers numbered (1) and (2) in the list below. Separate sessions of the Society were held on Wednesday afternoon and on Thursday morning and afternoon. On Tuesday evening several members took advantage of an invitation to attend the dinner and smoker of the Association of mathematical teachers in New England. The annual dinner of the Society took place on Wednesday evening, forty-seven members gathering for this agreeable occasion. Much credit for the success of the meeting must be given to the local committee on arrangements, Professors Tyler, Bartlett, and Bouton.

The total attendance at the annual meeting included the following sixty-one members:

Dr. C. S. Atchison, Professor D. P. Bartlett, Dr. E. G. Bill, Professor G. D. Birkhoff, Professor C. L. Bouton, Professor E. W. Brown, Dr. J. E. Clarke, Mr. G. R. Clements, Professor F. N. Cole, Professor L. L. Conant, Professor J. L. Coolidge, Mr. C. H. Currier, Mr. F. F. Decker, Dr. F. J. Dohmen, President E. A. Engler, Professor T. C. Esty, Professor F. C. Ferry, Professor W. B. Fite, Professor W. A. Garrison, Miss A. B. Gould, Professor F. L. Griffin, Professor J. N. Hart, Professor C. N. Haskins, Professor L. A. Howland, Professor E. V. Huntington, Dr. L. C. Karpinski, Professor O. D. Kellogg, Professor C. J. Keyser, Professor Gaetano Lanza, Dr. D. D. Leib, Dr. N. J. Lennes, Mr. Joseph Lipke, Dr. J. V. McKelvey, Professor H. P. Manning, Professor W. H. Metzler,

Professor G. A. Miller, Professor J. A. Miller, Mr. H. H. Mitchell, Professor W. A. Moody, Professor C. L. E. Moore, Professor G. D. Olds, Dr. F. W. Owens, Dr. H. B. Phillips, Dr. Arthur Ranum, Professor R. G. D. Richardson, Mr. W. J. Risley, Professor E. D. Roe, Professor Mary E. Sinclair, Professor D. E. Smith, Miss M. E. Trueblood, Professor H. W. Tyler, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor H. S. White, Professor J. K. Whittemore, Professor D. T. Wilson, Professor E. B. Wilson, Dr. Ruth G. Wood, Professor T. W. D. Worthen, Mr. W. C. Wright, Professor Alexander Ziwet.

Ex-President H. S. White and Professor E. W. Brown occupied the chair alternately during the several sessions. The Council announced the election of the following persons to membership in the Society: Professor R. M. Barton, Dartmouth College; Dr. J. R. Conner, Johns Hopkins University; Miss Eva M. Smith, London, England. Nine applications for membership were received.

The reports of the Treasurer, Auditing Committee, and Librarian have recently appeared in the Annual Register. The membership of the Society has increased during the year from 601 to 618, including at present 58 life members. The number of papers presented at all meetings during the year 1909 was 149. The total attendance of members at the meettings was 311. The Treasurer's report shows a balance of $8003.78, of which $3581.70 is credited to the life-membership fund.* Sales of the Society's publications during the year amounted to $1748.90. The Library has increased to nearly 3300 volumes. A catalogue of the Library, corrected to January 1, 1910, has been issued as a separate publication.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor L. E. DICKSON,
	Professor J. I. HUTCHINSON.
Secretary,	Professor F. N. COLE.

*The total income of the Society since January 1, 1895, has been $56,299.22, of which $30,310.94 is credited to members' dues. Disbursements in the same period have been $49,412.11, of which $39,008.47 has been expended for printing the *Bulletin*, *Transactions*, and other publications of the Society. The total returns from publications have been $13,508.99. Editorial expenses have been $2750.15, administrative expenses $7653.14.

Treasurer, Professor J. H. TANNER.
Librarian, Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor E. W. BROWN,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1912,

Professor D. R. CURTISS, Professor J. C. FIELDS,
Professor L. P. EISENHART, Professor P. F. SMITH.

The following papers were read at this meeting:

(1) Professor F. L. GRIFFIN: "Certain tests comparing areas and other geometrical magnitudes."

(2) Professor G. A. MILLER: "Groups generated by two operators s_1, s_2 satisfying the equation $s_1 s_2^2 = s_2 s_1^2$."

(3) Dr. H. M. SHEFFER: "Total determinations of deductive systems with special reference to the algebra of logic."

(4) Professor R. G. D. RICHARDSON: "The Jacobi criterion in the calculus of variations and the oscillation of solutions of m linear differential equations of the second order with m parameters."

(5) Dr. J. V. MCKELVEY: "The groups of birational transformations of algebraic curves of genus five."

(6) Professor J. L. COOLIDGE: "The representation by means of circles of the imaginary elements of a three-dimensional domain."

(7) Dr. L. C. KARPINSKI: "Jordanus Nemorarius and John of Halifax."

(8) Mr. H. H. MITCHELL: "The subgroups of the collineation group of the finite plane, $PG(2, p)$."

(9) Professor W. H. JACKSON: "Differential and integral equations arising out of the theory of radiation."

(10) Professor G. D. BIRKHOFF: "The stable solutions of the problem of three bodies."

(11) Professor W. D. CAIRNS: "The solution of the Lagrange equation in the calculus of variations by means of integral equations."

(12) Dr. ARTHUR RANUM: "On the line geometry of r emannian space."

(13) Dr. H. F. MacNeish: "Linear polars of quantics which are completely reducible to the product of linear forms."

(14) Professor E. V. Huntington: "An elementary explanation of the precession of a gyroscope."

(15) Professor C. J. Keyser: "Relational groups."

(16) Professor Edward Kasner: "Thomson and Tait's theorem on conservative forces."

(17) Professor Edward Kasner: "Note on Lamé's families connected with dynamics."

(18) Dr. Arthur Ranum: "On Clifford parallels and Clifford surfaces in riemannian space."

Dr. Sheffer was introduced by Professor Osgood. In the absence of the authors, Dr. MacNeish's paper was read by Professor Veblen, and the papers of Professors Jackson, Cairns, Keyser, and Kasner were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this note Professor Griffin observes that the process used in an earlier paper * for comparing areas, lengths, etc., of central orbits, can be applied to the purely geometrical problem of making similar tests in many cases where two curves are given by either polar or rectangular equations or by differential equations of the first or second order.

2. In 1878 Cayley published a note on the groups which may be generated by two operators s_1, s_2 satisfying the equation $s_1 s_2 = s_2^2 s_1^2$ and observed that it is not possible to represent all the operators of such a group in the form $s_1^a s_2^\beta$ except when the group is cyclic. In 1905 Netto considered the same relation in *Crelle* and observed that either the orders of s_1, s_2 are equal to each other or the order of one of these operators is twice the order of the other. In a recent number of the *Quarterly Journal* Professor Miller extended these results, but did not give many general theorems. In the present paper he gives several fundamental theorems which are implied in the above relation, and by means of these he obtains the known results much more easily and also arrives at a number of new results. In particular he shows that either the first or second of the three generational relations $s_1^5 = s_2^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ given by

* Read December 30, 1908.

Netto is redundant. That is, the two conditions $s_1^5 = 1$, $s_1 s_2^2 = s_2 s_1^2$ imply that $s_2^5 = 1$.

The following are some of the theorems established by Professor Miller in the present article: If two operators satisfy the relation $s_1 s_2^2 = s_2 s_1^2$, their squares are of the same order as their product, and they generate a group whose commutator subgroup is generated by two conjugate commutators and whose commutator quotient group is cyclic. If both of these operators are of odd order, they generate a solvable group whose commutator subgroup is either cyclic or the direct product of two cyclic groups. If α belongs to exponent 7 with respect to a prime number p and if $\alpha + 1 \equiv \alpha^2 \bmod p$, then $p = 29$ and $\alpha = 24$. Similar characteristic properties are established for the primes 5, 11, 19 as incidental results. It is also proved that $(s_1 s_2^{-1})^{4n} = s_2^{-1}(s_1^6 s_2^{-6}) s_2$ whenever s_1, s_2 are both of odd order.

3. Basal determinations ("postulate definitions") of various deductive systems — for example, of "ordinary" algebra, of the "algebra of logic," and of geometry — in terms of conveniently chosen (or basal) operations or relations have been developed recently. The present paper discusses the problem of the *total* determinations of deductive systems, that is, the total sets of element classes, relations, and propositions that are possible for a given system. The paper is thus intended as an introduction to a general theory of systems.

In particular, Dr. Sheffer solves the problem for the deductive system called the algebra of logic. He finds the total set of relations (and operations) which can serve as basal relations (and operations) for that algebra. Of this total set, the relations and operations employed hitherto — "logical addition," "logical multiplication," and "inclusion" (Huntington); the "between"-relation (Kempe); and the O-relation (Royce) — are shown to be isolated cases.

4. The relation between the Jacobi criterion of the calculus of variations and the oscillation of the solution of the self-adjoint differential equation of the second order $(pu) + qu + \lambda ku = 0$ was the subject of a paper read by Professor Richardson at the summer meeting of the Society. In a second paper of a series to appear in the *Mathematische Annalen* the results have been

extended to the case of m linear differential equations

$$(p_i u_i')' + q_i u_i + (\lambda r_{i1}(x) + \cdots + \pi r_{im}(x))u_i = 0 \quad (i = 1, 2, \cdots, m)$$

with the m parameters $\lambda, \cdots, \pi$. These differential equations are considered as the Lagrange equations of a calculus of variations problem, the functions $u_i(x)$ satisfying the boundary conditions $u_i(0) = u_i(1) = 0$ and a certain quadratic condition or the quadratic and nm linear conditions. The Jacobi criterion in each case determines exactly the number n of oscillations of the function $u_i(x)$ in the interval 0, 1.

5. The purpose of Dr. McKelvey's paper was to find the normal curves of hyperspace of genus 5 and the forms of the plane curves into which they can be projected; also to find the groups of birational transformations under which they are invariant. The plane curves of this genus are in general sextics with 5 double points. They become nodal quintics when a g_3^1 exists. The equations of the sextics were obtained by means of the quadratic relations among their adjoint curves of order 3. The curve of genus 5 is the only one which is completely defined by the quadratic relations among its adjoints of order $n - 3$. It was shown that the transformations of the sextic are related to the linear transformations of a certain plane quintic, except when the quadratic relations among the adjoints are also invariant. The largest group obtained was of order 192.

6. In 1872 Laguerre showed how the imaginary points of the finite three-dimensional domain could be represented by means of real circles of positive or negative radius. Professor Coolidge's paper exhibited two other methods of circular representation which afford a better separation between conjugate imaginaries, and a simpler representation of the simplest point systems.

7. The introduction into Europe of the Hindu methods in arithmetic is closely associated by Moritz Cantor in his Geschichte der Mathematik with the names of Leonard of Pisa and Jordanus Nemorarius. The prominent place given by Cantor to Jordanus rightfully belongs to John of Halifax (Sacrobosco). Dr. Karpinski's paper presents some new material in regard to the Algorismus by Jordanus and some notes on copies

found in American libraries of the Algorismus by Sacrobosco, and emphasizes the important rôle played by John of Halifax in spreading the system of the Hindus, taught by the Arabs.

8. In Mr. Mitchell's paper the determination is made of the subgroups of the collineation group in three variables, when the coefficients of the transformations $\rho x_i' = \Sigma a_{ij} x_j$ $(i, j = 1, 2, 3)$ are integers reduced modulo p, p being a prime. The order of the group is $(p^2 + p + 1)(p^2 + p)p^2(p - 1)^2$.

The problem is treated geometrically, the $p^2 + p + 1$ sets of marks $(\alpha\beta\gamma)$, upon which the group may be represented as a permutation group, being regarded as the points of a finite plane. A self-conjugate subgroup consisting of all transformations with determinant unity exists if p has the form $3n + 1$. There are subgroups leaving fixed a point, a line, a triangle, and a conic. The only additional subgroups are the G_{360}, the Hessian G_{216}, the G_{168}, and two subgroups of the Hessian, G_{72} and G_{36}. The methods employed in the determination of the groups of the latter class apply equally well to the case of finite groups in the ordinary geometry.

9. Schuster has discussed the transmission of radiant heat when the isothermal surfaces are parallel planes and conduction is neglected. The problem was reduced by him to the solution of the following equation:

$$\left(1 + \frac{\partial}{\partial t}\right)\frac{\partial^2 E}{\partial \mu^2} = \frac{\partial E}{\partial t}, \tag{1}$$

where E is the total density of radiation at any point. The aim of the discussion was to describe only the broad features of the phenomena and the following assumptions were made: (i) E is independent of the time, (ii) the radiation flows only normally to the isothermal planes, (iii) the absorption is independent of the wave length.

In the present paper, Professor Jackson finds that the removal of restriction (i) gives the length of time necessary to establish the steady state to a given degree of approximation, while the removal of restrictions (ii) and (iii) leads respectively to the following equations, which are reducible to ordinary integral equations of the second kind with symmetric kernel:

$$\left(1 + \frac{\partial}{\partial t}\right)\left(1 - x^2 \frac{\partial^2}{\partial \mu^2}\right) E = \int_0^1 E dx, \tag{2}$$

where t, x, μ are independent variables,

$$(3) \qquad \left(1+\frac{\partial}{\partial t}\right)\left(1-f(\kappa)\frac{\partial^2}{\partial m^2}\right)E=\int_0^1 Ed\kappa,$$

where t, κ, m are independent variables.

10. Professor Birkhoff discusses the stable solutions of the problem of three bodies in the light of an earlier paper on stability presented by him at the Princeton meeting, September 14, 1909.

11. The method of integral equations is applied by Professor Cairns to the Lagrange problem: To determine $y_1, \cdots, y_n$ as functions of x so that they make

$$I=\int_a^b F(y_1', \cdots, y_n'; y_1, \cdots, y_n; x)dx$$

a minimum and at the same time satisfy given differential equations

$$\phi_i(y_1', \cdots, y_n'; y_1, \cdots, y_n; x)=0 \quad (i=1, 2, \cdots, m<n).$$

(This includes isoperimetric problems as a special case.)

The requirement that the second variation

$$\int_a^b Qdx$$

shall be positive, calls for the existence of functions satisfying the equations (given for simplicity's sake for $n=2$)

$$L_1(u)+\lambda k_1 u_1+\lambda'\left(\frac{\partial\phi}{\partial y_1}-\frac{d}{dx}\frac{\partial\phi}{\partial y_1'}\right)=0,$$

$$L_2(u)+\lambda k_2 u_2+\lambda'\left(\frac{\partial\phi}{\partial y_2}-\frac{d}{dx}\frac{\partial\phi}{\partial y_2'}\right)=0,$$

$$u_1'\frac{\partial\phi}{\partial y_1'}+u_2'\frac{\partial\phi}{\partial y_2'}+u_1\frac{\partial\phi}{\partial y_1}+u_2\frac{\partial\phi}{\partial y_2}=0,$$

where

$$L_i(u) \equiv \tfrac{1}{2}\left(\frac{d}{dx}\frac{\partial Q}{\partial u_i'} - \frac{\partial Q}{\partial u_i}\right).$$

A system H_{ij} of *generalized* Green's functions shows this set of equations to be equivalent to the system of integral equations

$$\phi_1(x) = \lambda \int_a^b \{H_{11}(x, \xi)\phi_1(\xi) + H_{12}(x, \xi)\phi_2(\xi)\}d\xi + x_1' g_1(x),$$

$$\phi_2(x) = \lambda \int_a^b \{H_{21}(x, \xi)\phi_1(\xi) + H_{22}(x, \xi)\phi_2(\xi)\}d\xi + x_2' g_2(x),$$

$$\lambda' \int_a^b \Phi(x)dx + \lambda \int_a^b \Psi(\phi_1(x), \phi_2(x))dx = 0,$$

g_i, Φ, and Ψ being known functions.

Finally, the solution of these is reduced by a system of orthogonal functions

$$\psi_{11}(x), \quad \psi_{12}(x), \quad \cdots; \quad \psi_{21}(x), \quad \psi_{22}(x), \quad \cdots;$$

$$\int_a^b (\psi_{1i}^2 + \psi_{2i}^2)dx = 1, \quad \int_a^b (\psi_{1i}\psi_{1j} + \psi_{2i}\psi_{2j})dx = 0 \qquad (i \neq j)$$

to a generalized system of linear equations in an infinite number of variables already developed by the writer.

The theory is completely developed, with a full solution of the original minimum problem.

12. In this paper Dr. Ranum gives a classification of linear complexes, of congruences of order one and class one, and of quadric surfaces having real generators, in riemannian space. He also studies in detail three special classes of quadric surfaces, namely Clifford surfaces, surfaces of revolution, and normal surfaces (those having one generator of each set perpendicular to every generator of the other set).

13. By projective methods Dr. MacNeish obtains a recursion sequence of geometric constructions for the linear polar of a point as to a linear k-ad of points, as to a k-line in the plane, and in general as to a k-hedron in n-space. Then, using symbolic

notation, he shows analytically that the linear polars, obtained synthetically above, harmonize with the analytic polar theory for the n-ary k-ic which is the product of linear factors. A simple application of these results gives a construction for the linear polar of algebraic curves, surfaces, and spreads. A second application is the consideration of certain configurations from the standpoint of linear polarity. The quadrangle-quadrilateral configuration in the plane is generalized and a self dual configuration in n-space is obtained consisting of an $(n+2)$-point and an $(n+2)$-hedron. A second generalization gives an associated k-point and k-hedron in n-space which is reciprocal only for $k = n+2$. The invariantive conditions for reciprocity in the case of 4 collinear points is then considered and some interesting new geometric interpretations of the concomitants of the binary quartic form are obtained.

14. Professor Huntington's paper on the gyroscope obtains by elementary methods the actual accelerations of the several particles of the rotating disc when the axle of the disc is "precessing" with angular velocity ψ', and hence provides a direct elementary proof of the well-known fact that the applied forces required to maintain this precession constitute a couple lying in a plane perpendicular to the plane of precession and having a moment equal to $C\omega\psi'$, where C is the moment of inertia of the disc about its axle and ω the angular velocity of spinning. The following rule for the direction of precession is believed to be new: If the applied force be thought of as due to the pressure of a shelf against the axle, the precession will take place in the direction in which the axle would tend to roll along the shelf. The paper also calls attention to an error in a recent text-book, in regard to the motion induced by the applied couple in case precession is prevented. As a matter of fact, if the gyroscope is not allowed to precess, the rotation caused by the applied couple will be exactly the same as if the disc were not spinning.

15. From the postulates (as those of Russell, "Théorie générale des relations," *Revue de Mathématiques*, volume 7, page 115) of the calculus of relations, it readily follows that relations constitute an infinite class of potence equal at least to that of the continuum. Of the infinitude of classes of relations there is a large finite number of classes that by virtue of their

comprehensiveness and obviousness may be characterized as fundamental. Such are, for example, the classes known as symmetric, asymmetric, non-symmetric, transitive, intransitive, non-transitive, multiform, uniform, biuniform, and so on, together with the classes resulting from logical addition and multiplication of given fundamental relational classes. Professor Keyser's paper is a preliminary report upon the problem of determining which of the fundamental classes of relations are groups under such rules of combination (logical addition and logical multiplication, for example) as are applicable to relations without exception. Of several scores of relational classes thus examined, it is found that a goodly percentage possess the group property under one or both of the rules mentioned. The paper with detailed results will be published at a later date.

16. If particles are projected from an arbitrary point in all directions with a given speed and acted upon by any field of force, a doubly infinite system of trajectories will be obtained. Professor Kasner shows that only in the conservative case will these curves admit orthogonal surfaces. A second peculiarity of conservative fields is the mutual orthogonality of the three circles of curvature having four-point contact. The main part of the paper relates to a question suggested by the general theorem of Thomson and Tait: With what speeds must particles be projected normally from an arbitrary surface, so that the trajectories described shall form a normal congruence? In general the only solution is that for which the sum of the potential and kinetic energies is constant. For certain exceptional surfaces however other laws are possible.

17. In his second paper Professor Kasner determines those conservative forces for which every system of surfaces of equal action (in the sense of Thomson and Tait) constitutes a family of Lamé, that is, part of a triply orthogonal system. The trajectories are found to be circles. The result may be stated quite simply in optical terms and may be connected with non-euclidean geometry.

18. It is well known that the geometry of a congruence of Clifford parallels corresponds to the point geometry of a sphere. Some of the consequences that Dr. Ranum draws from this

correspondence are the following. If the distances between three Clifford parallels are a, b, c, and if the flux-angles between the three rectangular Clifford surfaces joining them are α, β, γ, then these six quantities are related to one another exactly as if their doubles were the sides and angles of a spherical triangle; e. g., $\sin 2a : \sin 2b : \sin 2c = \sin 2\alpha : \sin 2\beta : \sin 2\gamma$. The locus of a line that is right parallel to one of the generators of a cone of order m and left parallel to the others in order is a ruled non-developable surface S of order $2m$. Any curve drawn on the surface S and everywhere orthogonal to its generators will meet any generator in points whose distance apart is constant for the given surface and proportional to its volume.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-sixth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, December 31, 1909–January 1, 1910. Professor G. A. Miller, Chairman of the Section, presided at all of the sessions except at the opening on Friday morning, when Professor E. B. Van Vleck, Vice-President of the Society, occupied the Chair. The attendance at the various sessions included sixty-one persons among whom were the following forty-seven members of the Society:

Professor C. H. Ashton, Mr. W. H. Bates, Professor G. A. Bliss, Professor Oskar Bolza, Professor J. W. Bradshaw, Professor W. H. Bussey, Dr. Thomas Buck, Professor D. F. Campbell, Professor D. R. Curtiss, Professor J. F. Downey, Dr. Arnold Dresden, Mr. E. B. Escott, Mr. Meyer Gaba, Professor E. D. Grant, Mr. T. H. Hildebrandt, Professor F. H. Hodge, Professor T. F. Holgate, Professor Kurt Laves, Dr. A. C. Lunn, Mr. E. J. Miles, Dr. W. D. MacMillan, Dr. H. F. MacNeish, Professor G. A. Miller, Professor E. H. Moore, Professor C. N. Moore, Dr. R. L. Moore, Professor J. C. Morehead, Professor F. R. Moulton, Professor Alexander Pell, Mrs. Anna J. Pell, Miss Ida M. Schottenfels, Professor G. A. Scott, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Professor A. W. Smith, Professor A. L. Underhill, Professor E. B. Van

Vleck, Dr. G. E. Wahlin, Professor E. J. Wilczynski, Professor R. E. Wilson, Professor B. F. Yanney, Professor A. E. Young, Professor J. W. Young, Professor J. W. A. Young.

On Friday evening forty members of the Society dined together at the Quadrangle Club, and enjoyed one of the most interesting occasions in the history of the Section, in the way of social intercourse and the promotion of acquaintance and good fellowship.

At the business meeting on Saturday morning the following officers of the Section were elected for the ensuing year: Professor L. E. Dickson, Chairman, Professor H. E. Slaught, Secretary, and Professor W. B. Ford, third member of the program committee. At this time also the following resolution was introduced by Professor Van Vleck and carried by unanimous vote: Resolved that the Chicago Section respectfully requests the Council of the Society to arrange, in accord with the powers granted to it in By-Laws III and VII, that the next annual meeting of the Society be held with the Chicago Section, and that the President's address be also there given.

The following papers were read at this meeting:

(1) Professor C. H. ASHTON: "A new elliptic function."

(2) Professor C. N. MOORE: "On the uniform summability of the developments in Bessel functions of order zero."

(3) Miss HAZEL H. MACGREGOR: "Three-dimensional chains and a classification of the collineations in space."

(4) Mr. E. B. ESCOTT: "Logarithmic series."

(5) Mr. E. B. ESCOTT: "Calculation of logarithms."

(6) Professor E. J. WILCZYNSKI: "On the problem of three bodies."

(7) Dr. A. C. LUNN: "An abstract definition of limit."

(8) Dr. A. C. LUNN: "Note on the existence of the instantaneous axis in a rigid body.

(9) Mrs. ANNA J. PELL: "On an integral equation with an adjoined condition."

(10) Professor G. R. DEAN: "Generalized plane stress."

(11) Mr. W. H. BATES: "The medium curvatures of R_n in S_{n+1}."

(12) Dr. G. E. WAHLIN: "On the base of a relative field, with an application to the composition of fields."

(13) Professor D. R. CURTISS: "Note on a method of determining the number of real branches of implicit functions in the neighborhood of a multiple point."

(14) Professor E. B. VAN VLECK: "A functional equation for the sine."

(15) Professor E. B. VAN VLECK: "On certain extensions of Abel's functional equations and their relation to Weierstrass's algebraic addition theorem."

(16) Professor G. A. MILLER: "Groups generated by two operators each of which is transformed into a power of itself by the other."

(17) Professor W. A. MANNING: "The limit of the degree of primitive groups."

(18) Professor C. H. SISAM: "On three-spreads satisfying four or more linear partial differential equations of the second order."

(19) Dr. L. I. NEIKIRK: "Groups of rational fractional transformations in a general field."

(20) Professor J. W. YOUNG: "On the discontinuous zeta groups defined by the rational normal curves in a space of n dimensions."

(21) Dr. R. L. BORGER: "On the Galois group of the reciprocal sextic equation."

(22) Professor F. R. MOULTON: "The singularities of the solution of the two-body problem for real initial conditions."

(23) Professor J. B. SHAW: "On hamiltonian products."

(24) Mr. A. R. SCHWEITZER: "On the geometry of the projective line."

(25) Mr. A. R. SCHWEITZER: "On the dimensional extension of Grassmann's extensive algebra" (preliminary report).

The papers of Dr. Neikirk and Miss MacGregor were presented by Professor J. W. Young. In the absence of the authors the papers of Professors Dean, Manning, and Shaw, Dr. Börger, and Mr. Schweitzer were read by title.

Besides the above papers, informal reports from two committees of the International Commission on the teaching of mathematics were presented, one by Professor D. R. Curtiss on "Courses of instruction in universities," and one by Professor E. B. Van Vleck on "Preparation of instructors for colleges and universities." Considerable interesting discussion followed these reports.

Abstracts of the formal papers follow below. The numbering corresponds to that of the titles in the list above.

1. In Professor Ashton's paper, a set of four functions $O(z)$

is defined by the functional equation

$$O(z) = [1 - ae^{\frac{2\pi i}{w_1}(z+w_2)}]\, O(z + w_2),$$

under the condition that

$$\lim_{n=\infty} O(z + nw_2) = 1.$$

From this definition the developments of the function $O(z)$ as an infinite product and as an infinite series are determined and some of its properties are studied. Its development in another infinite product by means of Weierstrass's theorem is also obtained and from this it is simply expressed as an infinite product of gamma functions. Its multiplication theorems are determined. Its relation to the σ or θ function is shown to be similar to the relation of the reciprocal of the gamma function to the sine, and finally it is shown that all of the elliptic functions and the constants which occur in a discussion of these functions can be expressed very simply in terms of O functions.

2. In this paper Professor Moore establishes the following theorem: If the function $f(x)$ is finite and integrable in the interval $c \leqq x \leqq 1$, and has a derivative that is finite and integrable in the interval $0 \leqq x \leqq c$, where c is any positive constant < 1, then the development of $f(x)$ in Bessel functions of order zero will be uniformly summable to the value of $f(x)$ throughout the interval $0 \leqq x < x_0$, where x_0 is any positive constant $< c$.

It has already been shown by Professor Moore in a previous paper * that if $f(x)$ is finite and integrable in the interval $0 \leqq x \leqq 1$, the development will be uniformly summable to $f(x)$ in any closed interval lying in the interval $0 \leqq x < 1$, which does not include a point of discontinuity of the function and does not include the origin. The behavior of the series in the neighborhood of the origin was not discussed in the previous paper.

3. The classification of the collineations in space, on the assumption that two collineations are equivalent if one can be transformed into the other by a linear homogeneous transformation with complex coefficients, is well-known. If, however,

* *Transactions Amer. Math. Society*, vol. 10 (1909), pp. 391–435.

the coefficients are restricted to real numbers, the number of types of collineations will be increased. This classification brings up the important additional problem as to the conditions under which a collineation with complex coefficients can be transformed into one with real coefficients.

In a recent paper * Professor J. W. Young has considered these two problems for the complex line from the point of view of projective geometry, the notion of a linear chain being fundamental. In a subsequent paper,† by making use of the idea of a two-dimensional chain, Professor Young considered these problems for the complex plane.

Miss MacGregor applies the same principle of classification to the non-singular collineations in a complex space of three dimensions and considers the corresponding problems. She gives the classification of the collineations in space into nineteen distinct types, each of which leaves a three-dimensional chain invariant. Any such collineation may be represented with real coefficients. The necessary and sufficient conditions that a collineation be of this type are derived and the corresponding systems of invariant chains are determined.

4. The first paper by Mr. Escott completes the one read by him in April, 1904, before the Chicago Section. In the ordinary logarithmic series

$$\log\frac{X+d}{X-d}=2\left[\frac{d}{X}+\frac{1}{3}\left(\frac{d}{X}\right)^3+\cdots\right]$$

the problem is to express X in the form of polynomials in x so that both $X+d$ and $X-d$ shall have rational linear factors. The solution of this problem gives interesting applications of the elementary theory of numbers. A number of series of this kind have been developed by others, but there has been no systematic attempt to solve the problem. Mr. Escott shows how an indefinite number of series may be obtained where X is of degree 1 to 7, and gives four examples where X is of degree 10, some of the factors in the latter case being quadratic.

5. In his second paper Mr. Escott shows the application of the series in the preceding paper to the computation of loga-

* "The geometry of chains on a complex line," *Annals of Mathematics*, vol. 11 (1909), pp. 33–48.

† Read before the Society at its April meeting, 1908.

rithms. Huyghens has given a list of numbers differing by unity, having small factors, which may be used to calculate the logarithms of numbers as far as 100. This table is improved and extended to 200.

6. The problem of three bodies has been almost exclusively studied from the analytical point of view. In the present paper, Professor Wilczynski formulates a number of questions suggested by geometry, some of which are capable of direct and final answers, while others of a more difficult character are merely indicated. It is intended that all of these problems be subjected to a more thorough investigation in the future.

The center of mass of the three bodies is supposed to be at rest. Each of the masses will describe a certain curve, the straight lines joining them in pairs will generate three ruled surfaces, the plane of the three masses will envelop a cone. In this first paper the linear differential equations are set up which characterize the projective differential properties of some of these loci. Particular attention is paid to the question: can the ruled surface generated by the straight line joining two of the bodies be developable? It is found that in order that this may be so, the mutual distances must satisfy a certain differential equation of the second order, but the question of the compatibility of this equation with those of the problem of three bodies is provisionally left open. Can this developable be a cone? Leaving aside the trivial cases in which the cone degenerates into a plane or a straight line, whose existence is obvious, it turns out that the vertex of the cone must be at infinity, i. e., the cone is necessarily a cylinder. Moreover, the triangle formed by the three bodies must in that case constantly remain isosceles, the two equal sides being those which join the two bodies which are situated upon the same element of the cylinder to the third. It is then shown that there actually exist solutions of the problem of three bodies for which the triangle always remains isosceles, and that some of these give rise to the "cylindrical solutions" just indicated. The third body, in the case of such a cylindrical solution, always describes a plane curve similar to the corresponding plane section of the cylinder. It does not appear that the plane section of the cylinder can be determined in general as a simple curve, but the differential equations which characterize it projectively are obtained.

Another question which receives a simple answer is that as

to the existence of solutions in which one of the bodies describes an asymptotic curve upon one of the ruled surfaces generated by the line which joins it to one of the other bodies. The criteria for plane orbits are also indicated.

It is proposed to found a new general theory of perturbations upon a combination of the theory of osculating conics, cubics, and quartics of plane and space curves, with the method of the variation of constants.

7. In his first paper Dr. Lunn gives a general abstract setting of the concept of limit, designed so as to exhibit as particular cases the ordinary special limiting processes of analysis. The dependent variable is assumed to be numerical, real or n-dimensional complex, but the independent variable is taken as an abstract set of elements, undefined except as it is subject to an order relation satisfying certain postulates. The abstract theory, based on the abstract analogue of the proposition of Du Bois-Reymond, is devoted primarily to several theorems relating to necessary and sufficient conditions for interchange of two limiting processes. These yield as special cases such theorems as those on the integration and differentiation of infinite series, the differentiation of a definite integral, and the behavior of the solutions of differential equations as functions of a parameter.

8. In his second paper Dr. Lunn defines a rigid body with fixed origin as a non-coplanar set of points each at a constant distance from the origin and such that the scalar product of the vectors from the origin to any two points is constant. A purely analytic vector proof is given that there exists a single vector w such that the velocity of any point is given by $v = w \times r$.

9. Mrs. Pell shows that the solution of an orthogonal integral equation with an adjoined condition depends on the solution of an orthogonal equation without any condition.

10. In the case of an elastic plate under the action of a system of forces parallel to the faces and uniformly distributed across the edges, Professor Dean finds that the cubical expansion is a linear function of the two coordinates parallel to the faces; the coefficients being arbitrary constants which are determined by means of given surface tractions or displacements. This function when introduced into the equations of equilibrium

in terms of the displacements reduces the system to three independent equations, two of which are of Poisson's form in two dimensions, the third having a single term and being satisfied by the linear function which expresses the cubical expansion.

The solutions of the Poisson equations are expressed as surface and line integrals. The results substituted in the generalized form of Hooke's law give the normal components of stress and the components of shear.

Using Hertz's method of finding the pressure between two bodies in contact, thus determining the surface tractions and displacements, we have a complete and workable solution of the problem of determining the stresses in riveted and pin joints and in other problems of technical mechanics.

Errors frequently made in applying the theory of elasticity to technical mechanics, chiefly in assigning boundary conditions, are pointed out and means of avoiding them are suggested.

11. At any point of a surface in ordinary space, the curvatures of the two lines of curvature are the roots ρ_1 and ρ_2 of the equation

$$\begin{vmatrix} D + E\rho & D' + F\rho \\ D' + F\rho & D'' + G\rho \end{vmatrix} = 0 .$$

After division by the coefficient of ρ^2, the absolute term of this equation is the gaussian curvature of the surface, and the coefficient of ρ its medium curvature.

Similarly, at any point of a hypersurface R_n in the euclidean space S_{n+1}, the curvatures of the n lines of curvature are the roots $\rho_1, \cdots, \rho_n$ of the equation

$$(1) \qquad \begin{vmatrix} \alpha_{11} + a_{11}\rho & \alpha_{12} + a_{12}\rho \cdots \alpha_{1n} + a_{1n}\rho \\ \alpha_{21} + a_{21}\rho & \alpha_{22} + a_{22}\rho \cdots \alpha_{2n} + a_{2n}\rho \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_{n1} + a_{n1}\rho & \alpha_{n2} + a_{n2}\rho \cdots \alpha_{nn} + a_{nn}\rho \end{vmatrix} = 0 ,$$

in which the a's are the first fundamental quantities and the α's the second.

Equation (1) is of the nth degree in ρ and may be written

$$(2) \qquad \rho^n + K_1\rho^{n-1} + \cdots + K_{n-1}\rho + K_n = 0 ,$$

in which K_n is the Kronecker-Gaussian curvature of the hyperspace, and has been fully treated. In this paper, Mr. Bates studies the other coefficients of (2), which are called the medium curvatures of R_n in S_{n+1}.

12. If K is an algebraic field of degree N and k a subfield of degree n, Dr. Wahlin's paper shows that there exist in K $N/n = r$ numbers $\Omega_i (i = 1, 2, \cdots, r)$, and in k r ideals O_i, such that every integer in K can be expressed in the form

$$\sum_{i=1}^{N} \mu_i \Omega_i,$$

where the μ_i are integers of O_i, and moreover all numbers so represented are integers in K. It is then shown that the relative discriminant of K is equal to

$$O_1^2 \cdot O_2^2 \cdots O_r^2 \cdot |\Omega_i^{(k)}|^2,$$

where the index k is used to designate the relative conjugates to Ω_i.

These results are then used to deduce an expression for the discriminant of a field compounded from two fields in the case where the degree of the compounded field is equal to the product of the degrees of the two fields divided by the degree of the greatest subfield common to these two fields.

13. In a paper read before the Society in April, 1908, Professor Curtiss developed a method for determining the number of real branches of an implicit function $f(x, y) = 0$ in the neighborhood of a multiple point. It was there shown that if the point (x_0, y_0) is of multiplicity n, and if $\partial^n f(x_0, y_0)/\partial y^n \neq 0$, then the number of real branches passing through that point depends on the number of changes of sign in the values of $f(x, y)$ computed on each real branch of the implicit function $\partial f(x, y)/\partial y = 0$ in the neighborhood of (x_0, y_0). In the present note it is shown that if a_2 and b_2 have any but a certain finite number of values the implicit function $a_2 f(x) + b_2 f(y) = 0$ can be used in place of $f(y) = 0$, the restriction $\partial^n f(x_0, y_0)/\partial y^n \neq 0$ being thereby removed and other simplifications effected.

14. Cauchy has shown that the only real continuous solutions of the functional equation

$$\phi(x + y) + \phi(x - y) = 2\phi(x)\phi(y)$$

(other than the trivial solutions $\phi(x) \equiv 0$ and $\phi(x) \equiv 1$) are $\sin cx$ and $\sinh cx$. Professor Van Vleck gives in his paper a functional equation defining *uniquely* the sine function, and from this equation the properties of the function follow with great ease and rapidity.

15. In one of his earliest published papers Abel showed that if a differentiable function $f(x, y)$ possesses the property that $f[x, f(y, z)]$ is symmetric in x, y, z, then there exists a function $\phi(x)$ such that

$$\phi(f[x, y]) = \phi(x) + \phi(y). \tag{1}$$

Professor Van Vleck extends the theorem to a class of multiform functions $f(x, y)$ and derives a necessary and sufficient condition that an algebraic equation $G[\phi(x + y), \phi(x), \phi(y)] = 0$ shall present an actual (and not an impossible) addition theorem. Two generalizations of (1) were also discussed, the first of which has the form

$$\phi(f[x, y]) = f_1[\phi(x), \phi(y)].$$

16. Two special cases of the groups generated by two operators each of which is transformed into a power of itself by the square of the other have been considered in earlier papers by Professor Miller; namely, when the square of each of the two generators transforms the other generator either into itself or into its inverse. The object of the present paper is to obtain some fundamental theorems relating to the general case where the two operators s_1, s_2 satisfy the conditions

$$s_1^{-2} s_2 s_1^2 = s_2^{\alpha}, \quad s_2^{-2} s_1 s_2^2 = s_1^{\beta}.$$

If at least one of the two numbers α, β is even, the corresponding operator is of odd order and hence it must be generated by its square. In this case the group G generated by s_1, s_2 may be generated by a cyclic group and an operator transforming this cyclic group into itself. As many properties of these groups are well known, Professor Miller confines his attention to the cases in which both α and β are odd.

After observing that the subgroup H generated by s_1^2, s_2^2 is invariant under G and that $s_1^{2(\beta-1)}$, $s_2^{2(\alpha-1)}$ are invariant operators under G, it is proved that the orders of s_1, s_2 must divide $2(\alpha - 1)(\beta - 1)$. Hence each of these orders has an upper limit whenever α, β are both different from unity and only

then. It is proved that the fourth derived of G is unity and hence G is always solvable. It is also proved that the orders of s_1, s_2 are divisors of $2(\beta-1)^2$, $2(\alpha-1)^2$ respectively. The general theorems are illustrated by means of the special categories of groups which result when $\alpha=\beta=1$ and when $\alpha=\beta=5$. In the latter case the first derived group is abelian.

17. In *Liouville's Journal* for 1871 Jordan demonstrated the fundamental theorem:

"If a primitive group G (not containing the alternating group) contains a substitution A which displaces only m letters, the degree of G cannot exceed a certain limit."

It is a matter of great interest to reduce this limit as much as possible. In two subsequent papers* Jordan has given us expressions for the limit which are much lower than that first published. A further reduction† was recently made by Professor Manning, and the paper presented at this meeting gives a still lower limit. A special case of his general theorem may be stated thus:

The degree of a simply transitive primitive group which contains a substitution of prime order p on q cycles ($q<2p+3$) cannot exceed the greater of the two numbers $pq+q^2-q$, $2q^2-p^2$.

18. In this paper, Professor Sisam first determines under what conditions a three-spread in space of n dimensions can satisfy more than four linear partial differential equations of the second order. He then shows that if it satisfies four such equations, it has at each point four tangents having contact of the second order with the three-spread. The rest of the paper is devoted to the determination of the conditions under which two or more of these three-point tangents at a generic point may be consecutive.

19. This paper is an extension of that presented to the Chicago Section at the April, 1909, meeting by Dr. Neikirk. Several cases arise and the results of the former paper are extended to include these. The last part of the paper is devoted to finding a transformation $S^{(i)}$ which represents the product of the powers of several substitutions $S^{(i)}=R_1^{y_1^{(i)}}R_2^{y_2^{(i)}}\cdots R_k^{y_k^{(i)}}$.

* *Bulletin Soc. Math. de France*, vol. 1; *Crelle's Journal*, vol. 79.

† BULLETIN, vol. 13 (1907), p. 373.

$S^{(i)}$ is given by a polynomial in x and the exponents $y_r^{(i)}$ $(r = 1, 2, \cdots, k)$.

20. A rational normal curve C_n in a linear space S_n of n dimensions is left invariant by a three parameter group G of collineations $x_i' = \Sigma a_{ij} x_j$ $(i,j = 0, 1, \cdots, n)$ in S_n which is isomorphic with the group of all linear fractional substitutions on a single variable ζ. This isomorphism is most readily effected by interpreting ζ as the parameter of the points on C_n, so that to every collineation of G corresponds a unique ζ-substitution, and conversely. This fact has suggested a method of defining arithmetically certain discontinuous ζ-groups, a problem of the highest importance in the theory of automorphic functions. The method consists in determining G for a given C_n and then considering the discontinuous subgroup of G obtained by restricting the coefficients a_{ij} to be integers with determinant $|a_{ij}| = 1$. If this subgroup contains collineations other than identity, the C_n may be called an *integral* C_n, and the corresponding ζ-group will be discontinuous. That such integral C_n's exist follows at once from the fact that the *canonical* C_n, whose equations are $x_i = \zeta^{n-i}$ $(i = 0, 1, \cdots, n)$ gives rise in this way to the elliptic modular group.*

The case $n = 2$ of this method is the only one which has received detailed treatment.† By the consideration of a certain invariant J it follows that the discontinuous ζ-groups defined by the C_2's cannot contain elliptic substitutions of periods other than 2, 3, 4, and 6. By calculating the corresponding invariant for the general case C_n, Professor Young in a recent paper ‡ showed that the period e of any elliptic substitution occurring in a discontinuous ζ-group defined by a C_n must satisfy the relation

$$\sin\frac{(n+1)\pi}{e} = J \sin\frac{\pi}{e},$$

where J is any integer, positive, negative, or zero. In particular, the values $e = n$, $n+1$, $n+2$ satisfy this equation for $J = -1, 0, +1$. The present paper is devoted to the proof that integral C_n's really exist whose discontinuous ζ-groups con-

* J. W. Young, "On a class of discontinuous ζ-groups," etc., *Rendiconti del Circolo mat. Palermo*, vol. 23 (1907).

† Fricke-Klein, Automorphe Funktionen, v. 1, p. 533.

‡ BULLETIN, vol. 14 (1908), p. 367.

tain elliptic substitutions of periods n and $n+1$. The method of proof is direct in that it shows how to obtain the equations of a C_n whose group contains a given substitution of period n or $n+1$.

21. In this paper Dr. Börger finds the Galois group of the reciprocal sextic equation

$$x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 = 0$$

for the domain $R(1)$ of the coefficients. The group is a subgroup of the primitive substitution group G_{48}^6 having three systems of imprimitivity. Criteria for the irreducibility of the equation are also found.

22. The differential equations which are satisfied by the relative motion of two bodies subject to the newtonian law of gravitation define, when the initial values of the dependent variables are given, certain analytic functions of the independent variable t. The position and character of the singularities of these functions depend upon the properties of the differential equations and the numerical values of the initial conditions. In Professor Moulton's paper the positions of the singular points are found for all possible real initial conditions, their changes of position are determined as the initial values of the dependent variables are varied, the character of the singularities is found in all cases, the Riemann surfaces are constructed, and (except in a certain degenerate case) a variable is defined as a function of t in terms of which the coordinates can be expanded as power series convergent for all real values of t.

23. Professor Shaw's paper is in abstract as follows: Let there be a system of symbols, $\alpha, \beta, \gamma, \cdots$ either finite or infinite in number. The product of two of these, as α, β, is the symbol $\alpha\beta$, and is of the second rank. The product of $\alpha_1, \cdots, \alpha_n$ is $\alpha_1\alpha_2\cdots\alpha_n$, and is of rank n. These products may be associative or not, but they are distributive; that is $(l\alpha + m\beta)\gamma = l\alpha\gamma + m\beta\gamma$, etc., where l, m are ordinary numbers. If we restrict the product further we give it a special name and prefix a symbol, as $S\cdot\alpha\beta\gamma$, $V\cdot\alpha\beta\gamma$, from quaternions. These are called partial products.

The first partial product is defined by $I \cdot \alpha\beta$, which is such that it is a scalar, and

$$I \cdot \alpha\beta = I \cdot \beta\alpha;\ I \cdot (m\alpha)\beta = mI \cdot \alpha\beta,\ I \cdot (\alpha + \beta)\gamma = I \cdot \alpha\gamma + I \cdot \beta\gamma.$$

Also

$$\begin{aligned} I \cdot \alpha_1\alpha_2 \cdots \alpha_{2m} = {} & I\alpha_1\alpha_2 \cdot I \cdot \alpha_3 \cdots \alpha_{2m} - I \cdot \alpha_1\alpha_3 \cdot I \cdot \alpha_2\alpha_4 \cdots \alpha_{2m} \\ & + I \cdot \alpha_1\alpha_4 I \cdot \alpha_2\alpha_3\alpha_5 \cdots \alpha_{2m} \cdots \\ & + (-1)^i I \cdot \alpha_1\alpha_i \cdot I \cdot \alpha_2 \cdots \alpha_{i-1}\alpha_{i+1} \cdots \alpha_{2m} + \cdots \\ & + \cdot I \cdot \alpha_1\alpha_{2m} I\alpha_2 \cdots \alpha_{2m-1}. \end{aligned}$$

These forms turn out to be Pfaffians, and are such that

$$I \cdot \alpha_1 \cdots \alpha_{2m} = I \cdot \alpha_2 \cdots \alpha_{2m}\alpha_1 = I \cdot \alpha_{2m} \cdots \alpha_1.$$

Next we define the alternate product $A_m \cdot \alpha_1 \cdots \alpha_m$, such that $A_m \cdot \alpha_1 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_m = -A_m \cdot \alpha_1 \cdots \alpha_j \cdots \alpha_i \cdots \alpha_m$. It is of rank m. The Joly products are then defined by the statements

$$A_i \cdot \alpha_2 \cdots \alpha_{i+2h} = \Sigma \pm A_i \cdot \alpha_{j_1}\alpha_{j_2} \cdots \alpha_{j_h} \cdot I \cdot \alpha_{j+1} \cdots \alpha_{j_{i+2h}},$$

where $j_1 < j_2 \cdots < j_i$ and $j_{i+1} < j_{i+2} \cdots < j_{i+2h}$ and the $i + 2h$ subscripts are permuted in all possible ways, the sign being $+$ or $-$ according to the number of inversions.

Again

$$\begin{aligned} A_{h+i-2c}(A_h \cdot \alpha_1 \cdots \alpha_h A_i \cdot \beta_1 \cdots \beta_i) = {} & \Sigma \pm A_{h+i-2c} \cdot \alpha_{j_1} \\ & \cdots \alpha_{j_{h-c}}\beta_{k_1} \cdots \beta_{k_{i-c}} \cdot I(A_c\alpha_{j_{h-c+1}} \cdots \alpha_{j_h} A_c\beta_{k_{i-c+1}} \cdots \beta_{k_i}). \end{aligned}$$

(Example: $A_1 \cdot A_2\alpha_1\alpha_2 A_3\beta_1\beta_2\beta_3 = \beta_1 I \cdot A_2 \cdot \alpha_1\alpha_2 A_2 \cdot \beta_2\beta_3 - \beta_2 I \cdot A_2 \cdot \alpha_1\alpha_2 A_2 \cdot \beta_1\beta_3 + \beta_3 I \cdot A_2 \cdot \alpha_1\alpha_2 A_2 \cdot \beta_1\beta_2$.)

Finally, the Hamilton product of $\alpha_1, \cdots, \alpha_m$ is $H \cdot \alpha_1 \cdots \alpha_m = A_m \cdot \alpha_1 \cdots \alpha_m + A_{m-2} \cdot \alpha_1 \cdots \alpha_m + \cdots$, understanding that A_{m-m} or A_0 is I. These Hamilton products are associative, that is, for example, $H(H \cdot \alpha_1\alpha_2)(H \cdot \alpha_3\alpha_4\alpha_5) = H \cdot \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = H \cdot \alpha_1 H\alpha_2\alpha_3\alpha_4\alpha_5 =$ etc. The properties of these various expressions are then studied.

24. Mr. Schweitzer pointed out features of his axioms for the projective line analogous to his descriptive axioms for the

line (the system ${}^1R_1$) and for the plane (the system ${}^2R_2$). In particular, he showed how his planar descriptive system could be logically combined with linear projective axioms in such a way that the resulting system might be descriptive or projective, but not necessarily either.

25. In his second paper, Mr. Schweitzer presented a preliminary report on a study of the memoirs of F. Riesz * and H. Hahn † in relation to the generalization of Grassmann's extensive algebra ‡ to a denumerably infinite number of dimensions.

H. E. SLAUGHT,
Secretary of the Section.

THE SIXTY-FIRST MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE sixty-first meeting of the American Association for the Advancement of Science was held in Boston during the convocation week, December 27, 1909, to January 1, 1910. The president of the meeting was President D. S. Jordan, of Stanford University. The address of the retiring president, Professor T. C. Chamberlin, entitled "A geologic forecast of the future opportunities of our race" was given in Sanders Theatre, Cambridge, on the evening of the opening day.

Comparatively few papers on pure mathematics appeared on the separate program of Section A (mathematics and astronomy) because of the fact that the American Mathematical Society held its annual meeting in affiliation with the Association. The address of the retiring vice-president, Professor C. J. Keyser, of Columbia University, entitled "The thesis of modern logistic," was given on Wednesday morning at a joint session of Section A and the American Mathematical Society. At the same session Professor D. E. Smith presented a report on the work of the International Commission on the teaching of mathematics.

Another joint session was held on Tuesday afternoon under the auspices of the mathematicians and the physicists. During

* *Math. Naturw. Berichte aus Ungarn*, 1905, pp. 309, 341-343, etc.
† *Wiener Berichte*, Abt. II^A, vol. 116, pp. 601, 609-610, 642, etc.
‡ See also *American Journal*, October, 1909, pp. 365-410.

this session Professor C. Runge, Kaiser Wilhelm exchange professor of mathematics at Columbia University, read a paper "On the determination of latitude and longitude in a balloon," and Professor E. W. Brown read the first of his two papers in the list below. These joint sessions constituted the most noteworthy features of the program of Section A, and it is to be hoped that they may have tended to attract more investigators to the border land between mathematics and physics, where our country seems to be especially in need of workers.

The officers of Section A were: vice-president, E. W. Brown; secretary, G. A. Miller; councilor, G. B. Halsted; member of the general committee, H. W. Tyler; sectional committee, C. J. Keyser, E. W. Brown, G. A. Miller, E. O. Lovett, Harris Hancock, F. R. Moulton, Winslow Upton. On the recommendation of the sectional committee the following twenty-two members of the American Mathematical Society were elected fellows of the Association: R. P. Baker, W. E. Brooke, Thomas Buck, Arthur Crathorne, I. M. DeLong, C. E. Dimick, F. J. Dohmen, J. F. Downey, L. P. Eisenhart, J. C. Fields, B. F. Finkel, F. L. Griffin, A. G. Hall, C. N. Haskins, T. F. Holgate, J. I. Hutchinson, D. N. Lehmer, W. D. MacMillan, C. A. Noble, F. G. Reynolds, Charlotte A. Scott, A. W. Smith.

The following twenty papers were read before the Section:

(1) Professor C. J. KEYSER: "The thesis of modern logistic."

(2) Professor C. RUNGE: "On the determination of latitude and longitude in a balloon."

(3) Professor E. W. BROWN: "On certain physical hypotheses sufficient to explain an anomaly in the moon's motion."

(4) Professor D. E. SMITH: "The work of the International Commission on the teaching of mathematics."

(5) Mr. C. G. ABBOT: "The value of the solar constant of radiation."

(6) Professor F. W. VERY: "A new mode of measuring the intensities of spectral lines."

(7) Professor F. W. VERY: "The absorption of light by the ether of space."

(8) Professor F. W. VERY: "The fireball of October 7, 1909."

(9) Professor E. W. BROWN: "On a recent hypothesis and the motion of the perihelion of Mercury."

(10) Professor J. A. MILLER and Mr. W. R. MARRIOTT: "The heliocentric position of certain coronal streams."

(11) Professor D. P. TODD: "The mutual relation of magnifying power, illumination, aperture, and definition in telescopic work."

(12) Professor G. B. HALSTED: "La contribution non-euclidienne à la philosophie."

(13) Mr. H. E. WETHERILL: "Declination of the moon for Greenwich mean time."

(14) Mr. H. W. CLOUGH: "Meteorological waves of short period and allied solar phenomena."

(15) Professor MILTON UPDEGRAFF: "Recent work with the 6-inch transit circle of the United States Naval Observatory."

(16) Mr. V. M. SLIPHER: "Peculiar star spectra indicating selective absorption of light in space."

(17) Mr. R. M. STEWART: "Personality with the transit micrometer."

(18) Professor F. W. VERY: "Water vapor on Mars."

(19) Professor C. L. DOOLITTLE: "The existence of anomalous fluctuations in the latitude as shown by simultaneous observations with the zenith telescope and the reflex zenith tube of the Flower Observatory."

(20) Mr. LEON CAMPBELL: "Visual observations of variable stars at the Harvard College Observatory."

In the absence of their respective authors the papers by Professor Todd, Mr. Wetherill, and Mr. Slipher were read by title, and that of Mr. Stewart was read by Dr. O. J. Klotz. The remaining papers were read by their authors. Professor Keyser's vice-presidential address appeared in *Science*, December 31, 1909. Abstracts of the other papers of mathematical interest are given below. They bear numbers corresponding to those of the titles in the list given above.

2. Professor Runge remarked that the problem of finding the geographical position in a balloon from observations of the sun is very different from the same problem on a ship for the reason that in a balloon there is no dead reckoning. In a balloon the only way of getting the geographical position from the sun is by observing both altitude and azimuth at the same time. The accuracy with which the azimuth of the sun may be observed is rather rough; it would be difficult to obtain it within less than one tenth of a degree. Therefore the reduction of the observations need not be very accurate. At the

same time it is essential that the reduction should be made very quickly. For the time since the moment the observations were taken introduces an uncertainty which may be expressed by the area of a circle whose radius is equal to the distance through which the balloon may have traveled.

One naturally would therefore turn to graphical methods for the reduction of the observations. The reduction consists in finding the latitude ϕ and the hour angle t from the declination δ, the azimuth a, and the altitude h. Professor Runge proposes to find first the latitude ϕ from δ, a, h and then the hour angle t from δ, a, ϕ. In both cases we have to deal with the representation of an equation between four variables. Both of these equations may be written in the following form:

$$f(p) + h(r, s)g(q) = k(r, s),$$

where p, q, r, s denote the four variables. That is to say, two of the variables enter the equations in separate functions $f(p)$, $g(q)$, and the equation is linear in these functions, the coefficients being any functions of the other two variables. Equations of this kind may be represented graphically by the "méthode des points alignés" of Maurice d'Ocagne,* taking $f(p)$ and $g(q)$ as line coordinates, making $f(p)$ equal to the ordinate of the point of intersection of the straight line with the axis of ordinates and $g(q)$ equal to the gradient of the straight line, that is, the tangent of its angle with the axis of abscissas. In this way the rectangular coordinates of the point whose equation in line coordinates is the given equation become

$$x = h(r, s), \quad y = k(r, s).$$

For any given value of p the different values of q correspond to straight lines which form a pencil of rays whose center is on the axis of ordinates at the particular value defined by p, and any alteration of p would simply shift the center along the axis without altering the pencil of rays in any other way. The whole diagram may therefore be obtained by drawing two figures, one containing the curves r = constant and s = constant and the other containing the pencil of rays, and placing these two figures in the proper way one over the other. It so happens that the variable p is the declination of the sun which

* Traité de Nomographie.

may be regarded as constant during the ascent of the balloon. The aeronaut would therefore merely use a definite superposition of figures. These are photographed on transparent plates and a blue print is taken by copying the plates, one after another, on the same paper in proper position.

The aeronaut has one blue print to read off the latitude, a second to read off the hour angle after he has found the latitude. The equations are

$$\sin\delta + \cos\phi\cos h\cos a = \sin\phi\sin h,$$

$$\tan\delta + \sec\phi\sin t\cot a = \tan\phi\cos t.$$

In the first equation the curves $\phi = \text{const.}$ and $h = \text{const.}$ are the ellipses $x = \cos\phi\cos h$, $y = \sin\phi\sin h$. In the second equation the curves $\phi = \text{const.}$ and $t = \text{const.}$ are the confocal ellipses and hyperbolas

$$x = \sec\phi\sin t, \quad y = \tan\phi\cos t.$$

3. Newcomb has shown that there is a difference between the observed and the theoretical position of the moon which can be roughly represented by a term of period about 270 years and coefficient 13″. In the present paper Professor Brown examined numerous hypotheses sufficient to explain the term, in order to clear the ground of those which seemed to be of doubtful value and to bring forward those which appeared sufficiently reasonable to merit tests from observations of a different nature. He gave an account of three of these hypotheses, stating that a minute libration of the moon would be sufficient provided it took place in the moon's equator and had the proper period. The supposition of magnetic attraction practically demanded a periodic change in the magnetic moment of the earth or of the moon. If this were rejected, it was necessary to suppose that the mean place of the lunar magnetic axis was near the lunar equator and that the oscillations of its position took place in the plane of the equator. The observed secular change of the earth's magnetic axis could not produce the phenomenon without demanding a larger motion of the lunar perigee than observations warrant. On the border line between two sets of hypotheses is a curious fact, namely, that if the period of the solar rotation coincides very nearly with one of the principal lunar periods, a minute equatorial ellipticity of the sun's mass would be sufficient to explain the term. So

far as known these hypotheses do not conflict with any observed phenomena, but they cause some theoretical difficulties.

4. The International Commission on the teaching of mathematics was suggested some years ago but the first steps in its organization were not taken until April, 1908. At that time the Fourth International Congress of Mathematicians, then in session in Rome, empowered Professor Klein of Göttingen, Sir George Greenhill of London, and Professor Fehr of Geneva to appoint such a commission, and to arrange for it to report at the next congress to be held at Cambridge in 1912. As a result three commissioners have been selected from each of the leading countries and the work has actively begun. Those of the United States are Professors D. E. Smith, W. F. Osgood, and J. W. A. Young. It is expected that each of these countries will submit a very full report of the nature of the work in mathematics, from the kindergarten through the college, with some discussion of the range of advanced work in the universities. In the United States the investigation is carried on by means of fifteen committees, each divided into subcommittees. Professor Smith stated that about two hundred and seventy-five people are engaged in the work and the subcommittee reports will be submitted during the present winter. The committee reports will be submitted before the summer of 1910, and the national report by Easter, 1911. This paper is to appear in the *American Mathematical Monthly*.

9. Professor Brown's second communication consisted of a brief account of the hypotheses of Seeliger brought forward to account for the outstanding large residual in the motion of the perihelion of Mercury and the small residuals in the secular motions of the four minor planets. An analysis of the nature of the three hypotheses and a comparison of the number of arbitrary constants introduced, with the number of residuals to be accounted for, were also given.

10. Professor Halsted's memoir describes the meaning and growth of non-euclidean geometry, sketches its history and founders, and points out that philosophy has found a new criterion in this subject. The memoir is to appear in French in the *Mémoires de la Société des Sciences physiques et naturelles de Bordeaux*.

The next regular meeting of the Association will be held at the University of Minnesota under the presidency of Professor A. A. Michelson, of the University of Chicago. Professor E. H. Moore, of the University of Chicago, was elected vice-president and chairman of Section A; Professor G. A. Miller, of the University of Illinois, continues as secretary. Professor E. R. Smith, of the Brooklyn Polytechnic Preparatory School, was elected member of the sectional committee for five years.

G. A. MILLER,
Secretary of Section A.

SHORTER NOTICES.

Rara Arithmetica. By DAVID EUGENE SMITH, Teachers College, Columbia University. Second edition. Boston, Ginn and Company, 1909. xviii + 507 pp. Cloth. Price, $4.50.

THE title page of this splendid volume modestly states that "the work is a catalogue of the arithmetics written before the year 1601 with a description of those in the library of George Arthur Plimpton of New York." Another appropriate title might be, A brief history, on the bibliographical plan, of the genesis and content of sixteenth century arithmetic.

As a bibliography this work is more extensive than any of its predecessors, and is nearly complete for the formative period between 1472 and 1601. There are mentioned over five hundred and fifty different works, which number swells nearly to twelve hundred by the inclusion of the various editions. About four hundred and fifty of the different books are genuine arithmetics, while the others deal partially with algebra, astrology, or the calendar. In determining the significance of this number one thinks at once of De Morgan's Arithmetical Books, the best of the earlier authorities on the subject, and recalls that this work gives only seventy arithmetics printed before 1600. The list in Professor Smith's Rara Arithmetica is much more extensive than those of Graesse, Hain, and Coppinger, and contains more Italian titles than are given by Riccardi in his Bibliotheca Mathematica Italiana and more German ones than are included by Murhard in his Bibliotheca Mathematica.

But this work is more than a scholarly, well edited digest of all the earlier bibliographies; it is the result of the examination

of the original works in so far as they are extant and are to be found in the libraries of Europe or America. In particular, it is an extensively illustrated catalogue of the arithmetical collection of Mr. Plimpton. This collection is already well known to many scholars, and it will now become known to other students of mathematics wherever Professor Smith's work may circulate. It is the third great private collection of rare arithmetical books of the sixteenth century and surpasses its predecessors, the Libri and Boncompagni libraries, in respect to this period. It contains over three hundred arithmetics, while De Morgan with the aid of the British Museum was able to consult less than one hundred of those printed before 1601, including all editions. It is probable that Mr. Plimpton's library does not lack more than a dozen or so extant sixteenth century arithmetics that went through two editions.

The work is arranged chronologically by first editions, but this classification is supplemented by indices of reference, both alphabetical and geographical, making readily accessible every important fact in the volume. In describing a specimen, the first edition is always taken, if available, and discussed under these heads: title, colophon, description, editions. The description includes the size of the page and the text in centimeters; the number of blank, numbered, and unnumbered pages; the style of numbering; and the place of publication and date. The title pages of all the works of special value are reproduced in fac-simile, besides many pages and selections that show the evolution of arithmetic; and in addition to these there are twelve plates of particular interest to the student of early mathematical literature.

But Professor Smith's work is more than a catalogue, it is a condensed history of arithmetic during its formative period. The numerous notes not only point out the significant features of the specimens under discussion, but they form a comparative study of the subject from many points of view, for example, the comparison of the arithmetic of the Latin schools with that of the trade schools, the comparison of the arithmetics of different nations, and the relation of abacus reckoning to figure reckoning. The following excerpt illustrates this feature of the work:

"This work (Scheubel's Arithmetic) is the production of a scholar rather than a man conversant with the demands of business. While Scheubel tried to write a mercantile arithmetic, the result was far removed from the needs of the common people.

It carries the work in subjects like the roots so far that the ordinary Rechenmeister could not have used it. Moreover, it is written in Latin and is much more extended than the work of Gemma Frisius, so that it appealed neither to the business school nor to the ordinary classical school. A great deal of attention is given to exchange, the rule of three, and the extracting of roots of high order. Attention is also given to problems which would now form part of algebra, and there is a brief treatment of geometry from the standpoint of mensuration.

"While Scheubel is not much appreciated to-day, he was really ahead of his time. He tried to banish the expression 'rule of three' and to substitute 'rule of proportion.' His explanation of square root is in some respects the best of the century, and he dismisses with mere mention the 'duplatio' and 'mediatio' of his contemporaries. He extracts various roots as far as the 24th, finding the binomial coefficients by means of the Pascal triangle a century before Pascal made the device famous."

As to its usefulness, this is a work which no bibliographer of rare books will fail to consult. It will become an authoritative source for writers of mathematical history and the standard reference book on sixteenth century arithmetic for scholars in mathematics everywhere. It would be wasteful of the reviewer's space to speak of the author, because his special fitness is known to practically every student of the history of mathematics, and his scholarship stamps with authority all of his productions.

LAMBERT L. JACKSON.

Coordinate Geometry. By HENRY BURCHARD FINE and HENRY DALLAS THOMPSON. New York, The Macmillan Company, 1909. 8vo. 8 + 300 pp.

IT was generally considered by the writers of the earlier American text-books on analytical geometry and by those who then taught the subject that the material for a first course consisted of the chief metrical properties of the separate species of conic sections. There is a marked similarity between the text in these books and the easier portions of Chapters I, II, VI, X, XI, XII of Salmon's Treatise on Conic Sections (edition of 1869). Within recent years, however, there has been a marked tendency among some of the teachers to regard the acquisition of these isolated facts about parabolas, ellipses, and

hyperbolas as a means instead of an end. They are eager to infuse into even a first course some of the spirit of the so-called "modern" geometry and to replace a few of the less important facts of the old course by some of the simplest of the "new" ideas. Many of the recent texts have reflected more or less of this tendency. The advocates of the older method argue that it is necessary to master their facts before the newer ideas can be grasped. They also urge that their material, if thoroughly learned, will occupy all the time allotted; and that any attempt to introduce additional ideas will only result in lack of mastery and consequent confusion. The leaders of the innovation reply that the natural result of progress in mathematics, as in other subjects, is gradually to replace the less important old facts by the more important new ones. They also insist that some of the new ideas will tend to unify the mass of metrical facts.

In looking over a new book on analytical geometry, it is natural, therefore, to inquire whether there is any departure from the conventional body of material and whether there is any striking feature in the method of presentation. There was a strong tendency for some years to introduce into the elementary texts on analytical geometry, as well as those on calculus, many more or less complicated problems from various subjects, especially from physics and statistics. The discussion of loci and graphs gave an excuse for prolonged excursions into domains bordering on the mathematical territory. That a moderate use of such problems is stimulating to thought and is necessary to give some idea of the practical applications of mathematics is probably not often questioned. However, in some cases at least, this practice was undoubtedly overdone. In the present text almost the opposite extreme is found. There are practically no problems showing the applications of mathematics. In Chapter XI, on equations and graphs of certain curves, the path of a projectile is discussed. Perhaps there may be other applications, but they were not apparent in a first survey. There are places where such problems and illustrations are particularly useful. For example, in the first chapter, on coordinates, some illustrations of the use of coordinates and coordinate paper in the charts employed in many businesses to-day would add life to the chapter, without in any way lessening its dignity.

The authors state that it is in deference to usage that the chapter following that on the straight line is devoted to the

circle. They urge that it would be better to omit this until the chapters on the parabola and ellipse have been studied. Their reason is that thus "the student sooner realizes the power of the method of the coordinate geometry through seeing it employed in investigating *new* material." On the other hand, in justification of the retention of a short chapter on the circle in this place, it may be argued that, as the *method* is new, the student must gain some facility in its use by employing it on old material before he can have sufficient mastery of it and confidence in it to enable him to use it easily and naturally on the new material. In the treatment of the conics, either of two methods is possible. According to one, the discussion of conics in general precedes the briefer consideration of the particular properties of the different species. In the other method the order of presentation is reversed. There are, of course, arguments for and against each method. The second method is that employed in the present book, where Chapters IV, V, and VI deal with the parabola, ellipse, and hyperbola respectively, while conics in general are not discussed till Chapter VIII.

The treatment of tangents and poles and polars is always an interesting subject for discussion. One criticism that might be made on this book is that there is too much repetition in the subject of tangents. In the cases of the parabola and ellipse, three methods for finding the equation of the tangent in terms of the coordinates of the point of contact are worked out in detail. For the hyperbola the corresponding equation is derived from that for the ellipse. Would it not have been wiser for the authors to have selected one of these methods and omitted the other two? The teacher whose chief interest is in engineering students would probably wish the calculus introduced in the derivation of the tangent to the first conic. This, of course, brings up the ever old and ever new question of whether analytics shall be treated without the aid of calculus, or whether an attempt shall be made to combine the two subjects. As the present text-book is evidently intended for students who expect to have a course in calculus later, there is little use made of its symbols here. Nevertheless, they are not entirely neglected. In one of the chapters on solid geometry, after the equation of the tangent plane to a conicoid in terms of the coordinates of the point of contact has been obtained, the statement is made that the coefficients may be represented by the partial derivative symbols (provided the variables are replaced by the coordinates

of the point of contact). Again Table C at the end of the book is a one page explanation of derivatives and partial derivatives.

Poles and polars are not discussed in the separate chapters on the circle, parabola, ellipse, and hyperbola, nor in the chapter on the general equation of the second degree. They appear for the first time in Chapter IX, on tangents and polars of the conic. After the equation of the tangent to any conic in terms of the coordinates x', y' of the point of contact has been obtained, the statement is made that the straight line represented by this equation is called the polar of the point (x', y') with reference to that conic, whether point (x', y') lies on the conic or not. For the circle, it is shown that if the polar of P_1 contains P_2 then the polar of P_2 contains P_1. A solution is given for finding the pole of a given line with reference to a circle. It is also noted that the polar of a point outside a circle is the chord of contact for the tangents from that point. The corresponding fact if the point is inside the circle is stated. No mention is made of harmonic properties. It is to be noted that poles and polars hold a very insignificant place in the plan of this book.

In view of the recent discussions on the question of the place of loci in college entrance examinations, it is interesting to notice the treatment of this subject in an elementary text intended for the early part of a college course. In addition to the illustrations of loci found in the several conics and in their diameters and in certain of the special curves, the last chapter in the plane geometry is devoted to this subject. Nine examples are fully explained and then, after some general remarks on loci, fifty carefully graded problems are given. The idea of loci is certainly made prominent.

Solid geometry occupies more than one third of the volume. This is a more extended treatment than is usually found in the elementary texts. In the present book the first two chapters, on coordinates and direction cosines and on planes and straight lines, are very similar to corresponding chapters in C. Smith's Elementary Treatise on Solid Geometry. These two chapters occupy slightly less than one half of this part of the book. After a discussion of the general shape and the sections of the different species of conicoids and a brief survey of polar coordinates and transformation of coordinates, a chapter is devoted to the general equation of the second degree. Centers and diame-

tral and principal planes and the classification of conicoids are then briefly treated. There is a short discussion of the invariants of the general equation under a transformation from one orthogonal system of axes to another orthogonal system.

The text proper is followed by six short tables, which deal with algebraic and trigonometric formulas, derivatives and partial derivatives, four-place table of logarithms of a few numbers, lengths of arcs in radians, and the letters of the Greek alphabet. The book is concluded by a set of nine very good plates showing the silk thread figures of the ruled surfaces of the second order, as well as the usual plaster models of the conicoids. Indeed, one of the best features of the book is to be found in the excellence of the numerous figures. The young man in whose hands this text is placed will probably note first of all that it is small enough to fit into his pocket. By employing rather thin backs and paper that is not too heavy and by lessening the margins, the size and weight of the volume have been reduced to a minimum. Perhaps the strongest feature of the book is to be found in the abundant supply of examples. After each bit of theory there are some exercises, and at the end of each of the longer chapters there is a set of about fifty carefully graded problems.

E. B. COWLEY.

Leçons sur les Fonctions définies par les Equations différentielles du premier Ordre. Par PIERRE BOUTROUX. Paris, Gauthier-Villars, 1908. 190 pp.

THE little volume bearing the above title is one of the series of monographs on the theory of functions published under the editorship of E. Borel. The author's aim is to set forth the theory of functions defined by a differential equation as based on the work of Painlevé. He abandons the "local point of view" of Cauchy and studies the ensemble and form of the integral not only in the neighborhood of a point but in general. The particular question discussed is one raised by Painlevé, viz., how does the solution behave when the initial point x_0 at which it is considered varies from point to point in an arbitrary manner.

The book is divided into five chapters. Chapter I presents the fundamental notions. After a review of the usual theory of singular points the following theorem of Painlevé's is dem-

onstrated: As x varies from x_0 to $\bar{x}$ along a curve L joining x_0 to $\bar{x}$ an integral Y of the equation

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)} \tag{1}$$

(P and Q being polynomials in y) which takes in x_0 the value C will approach a definite value.

From this theorem the following one is deduced at once: Aside from fixed singular points, an integral of (1) can have only poles or algebraic critical points for singularities.

Another important theorem also due to Painlevé is given in this chapter: If P and Q are polynomials in y and algebraic in x and if the integrals are multiform functions of n branches then equation (1) can be reduced to Riccati's equation by a rational change of variable. One can always determine by a finite number of operations if this change of variable is possible. A discussion of multiform functions of an infinite number of branches and the corresponding Riemann surfaces closes the chapter.

Chapter II deals with the growth and behavior of a branch of an integral. The subject here discussed is the manner in which an integral increases as x increases. The problem is to determine whether an integral increases more slowly or more rapidly than a power of $|x|$ or than an exponential function. The subject is treated in quite a fascinating manner, but the presentation must be read in order to be appreciated.

Chapter III is on classification of singularities. Transcendental singular points are divided into two classes, those directly critical and those indirectly critical. A point x is directly critical if when an elementary circuit is described about x an infinity of branches are permuted. It is indirectly critical if an infinity of branches are permuted when an elementary circuit is described not about x properly speaking but about an infinity of critical points which are converging toward x as a limit. Following the above definitions the singular points of the equation

$$2z\frac{dz}{dx} = \alpha x + \beta z \tag{2}$$

are discussed in considerable detail under various assumptions, and the manner in which the branches permute themselves is shown.

After a discussion of other particular examples a few pages are devoted to a more minute classification of transcendental singularities.

Chapter IV considers the singular points of Briot and Bouquet. The equation which has singularities of this type is connected with the equation above by introducing a parameter μ which put equal to zero gives equation (2) and put equal to unity gives the equation under consideration. By allowing μ to vary, the intimate relation between the singularities of equation (2) and those of Briot and Bouquet is established.

Chapter V discusses some of the relations which exist between the singularities of the same equation.

The volume closes with a note of fifty pages by Painlevé: "On the differential equations of the first order whose general integral has only a finite number of branches."

C. L. E. MOORE.

Sur les premiers Principes des Sciences mathématiques. Par P. WORMS DE ROMILLY. Paris, A. Hermann, 1908. 8vo. 51 pp. 2.50 fr.

THIS essay undertakes to give an account of the recent work on the foundations of mathematics. The author concludes that the only branch of mathematics completely applicable to natural phenomena is arithmetic, since it depends solely upon the numeration of objects, and makes no hypothesis regarding their nature. Geometry, on the other hand, imposes upon them certain purely ideal hypotheses which indeed may differ so as to produce at least three systems of geometry, the system which nature is built upon being possibly that of Euclid, possibly otherwise. The contrast drawn here between the external validity of arithmetic as over against that of geometry is a little difficult to reconcile with the explanations devoted by the author to the varying systems of axioms on which arithmetic may be based. In fact he distinctly speaks of diverse systems of numeration. We might inquire, for example, are objects subject to the archimedean axiom or not?

A disproportionate amount of space is devoted to the setting forth of some seven foundations upon which geometry may be based, and not quite so much to mechanics. The reason for this is the underlying thesis which the author seeks to prove. He examines the different modes of grounding geometry and concludes they are all à priori and inapplicable to real objects

until certain other unprovable results of intuition are brought into play. Exactly what an external object consists of aside from its being a projection of an internal idea is not shown. And if the world as we conceive it is merely a projection of that which is wholly mental, then why so much struggling to prove the geometrical character of the world as we geometrize it? Or on the other hand, why such a certainty of its arithmetic as we arithmetize it?

The definition given by C. S. Peirce for mathematics has not been surpassed: "The study of ideal constructions (often applicable to real problems), and the discovery thereby of relations between the parts of these constructions before unknown." This implies the rôle of logic and of intuition in the architecture of this vast structure. And in a projection of two figures is A the projection of B, or B of A? Is the world framed according to the architecture or the architecture according to the world? Qui sait!

JAMES BYRNIE SHAW.

Taschenbuch für Mathematiker und Physiker. 1909. Von FELIX AUERBACH. Leipzig, Teubner. 1909. xliv + 450 pp. 6 Marks.

THIS little pocket manual initiates a series of year books to be issued by the firm publishing it. They are to be congratulated upon their enterprise in furnishing the mathematical public what it has long needed. The engineer has his Trautwine, Kent, Kidder, or Foster, but so far the mathematician has had only collections of integrals, or small collections of trigonometric formulas. This volume, on thin but opaque paper, with typography which is delightfully clear, contains not only an excellent summary of the whole field of mathematics, but also a resumé of mechanics, physics, and physical chemistry. One is much surprised and pleased at the amount of valuable material compressed into so small a space, yet so easily found. The chief formulas and definitions are to be found here for arithmetic, algebra, group theory, combinatory analysis, determinants, series, differential calculus, integral calculus, definite integrals, calculus of variations, differential equations, transformation groups, functions of a real variable, functions of a complex variable, gamma function, elliptic integrals and functions; principles of geometry, topology, planimetry, stereometry, goniometry, plane trigonometry, spherical trigonometry; coordinate

geometry, lines in a plane, general plane curves, conics, cubics, general space geometry, algebraic surfaces, families of surfaces, quadrics, twisted curves; line geometry, transformations in a plane; differential geometry of the plane and of space; probabilities, errors, numerical calculations, graphics, vector analysis and quaternions. The other headings mentioned are equally fully treated. Many topics have a place assigned for future exposition. In a few years this small encyclopedia will be almost a necessity for student, teacher, and investigator.

The book is illustrated with a portrait of Lord Kelvin, and his biography opens the introduction. The properly "year-book" topics are a calendar, astronomical data, lists of journals and proceedings and transactions of learned societies, new books, necrology for 1908, lists of teachers of mathematics and physics in Germany. The errors in the book are few, so far as the reviewer noticed in his reading; those existing are noticeable at once and doubtless will disappear in the next volume.

JAMES BYRNIE SHAW.

Statistique Mathématique. Par H. LAURENT. Paris, Octave Doin, 1908. vi + 272 + xii pp.

THE author states that, for him, the object of mathematical statistics is to indicate and investigate methods of making good observations, when the point in question is to make numerical estimates concerning matters which interest economists. He has thus limited his purposes to matters which relate to specific applications. This fact may account, in part, for the entire omission of that important body of mathematical statistics which has been developed in close connection with applications to biology. However, these methods have been applied by others to problems of economics.*

It is well stated in the preface that it is a very common error to suppose that those who direct statistical investigations do not need to know mathematics. The author remarks that official statistics are not good, in general, because those who direct statistical investigations are not prepared for the work, and that if it is not necessary to exact of the statistician that he have a command of universal science, it is necessary, at least, that he should have surveyed the field of scientific knowledge.

*See Yule, *Journal of the Royal Statistical Society*, vol. 60, pp. 812–854. Norton, Statistical Studies in the New York Money Market.

The book begins with the elements of probability — leading soon to the principle of Bayes on the probability of causes. In the proof of the inverse of the theorem of Bernoulli, it may be worth while to call attention to a few details which may confuse the reader: If l is to be used as indicated on page 28 as a deviation from a/s, instead of using l/s as on page 26 for this purpose, the expression $P = \theta(ls/\sqrt{2\alpha\beta s})$ on page 30 becomes $P = \theta(l\sqrt{s^3/2\alpha\beta})$. The expression $P = \theta(ls/\sqrt{2\alpha\beta s})$ is correct if l/s is used throughout as a deviation from a/s. On page 29, in the formula for P, the radical should be $\sqrt{s/2\alpha\beta\pi}$ instead of $\sqrt{\alpha\beta/2\pi s}$.

On page 37 there appears to be a numerical error such that we should read $\theta(l/628)$ instead of $\theta(l/6280)$ but the conclusion is all the more valid if this change is made.

In developing what he calls a general theory of errors, the author follows the well-known method of obtaining the normal curve from the hypothesis of Gauss in regard to the arithmetic mean. In presenting the method of least squares, the analysis of Laplace is followed rather closely, and the approximation consists in neglecting

$$\int_{-\infty}^{\infty} \phi(\epsilon)\epsilon^h d\epsilon \quad (h > 2),$$

where $\phi(\epsilon)d\epsilon$ is the probability (facilité) of an error between ϵ and $\epsilon + d\epsilon$, but no further assumption is made in regard to the form of the function $\phi(\epsilon)$. The integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{\mu(t-a)\sqrt{-1}}\frac{\sin \mu l}{\mu}\, d\mu$$

is used as an auxiliary function to evaluate the integral which expresses the total probability of the concurrence of errors.

The greater portion of the book is given to applications rather than to pure theory. The various sources of statistics in France are indicated, and conditions are discussed which must be satisfied in order that an observation which is known only by description may be of value. Agricultural statistics which report crops of grain to seven significant figures are severely criticised in the following terms: "As there is no cause to suspect the honesty of those who publish these figures, it is necessary to infer their ignorance in scientific matters or their native sim-

plicity, and then only natural to doubt the exactness of the first figures."

It is stated that of all statistics relative to man the best and perhaps the most important deal with the duration of human life and with immigration. The book contains the well known methods (aside from the method of moments) of graduating mortality statistics, with the somewhat reasonable à priori considerations which lead to the various types of functions used in such graduations.

An argument is given to establish that part of the law of Malthus

$$p = p_0 e^{k(t-t_0)}$$

which states that populations have a tendency to increase in geometrical progression, but of the part stating that subsistence tends to increase in arithmetical progression, the author says he does not see how this can be established.

In studying the assessment of taxes for Prussia, and also for other countries that have adopted the same method of collection, M. Pareto arrived at the following law: If N represents the number of individuals who have an income over x, then

$$N = Ax^{-\alpha}e^{-\beta x},$$

or less exactly (β being very small),

$$N = Ax^{-\alpha},$$

A, α, β being independent of x and of N.

It is stated that it is probable that this formula is applicable to all countries, and some arguments are given in favor of the law.

To the reviewer one of the most interesting sections of the book consists in a mathematical argument from which it results that under free competition prices are determined to the greatest satisfaction of all concerned in the transactions. This argument is based on the assumption that an individual retires from a market satisfied when a certain mathematical relation

$$\phi(q_1, q_2, \cdots, q_s) = a$$

exists between the quantities of merchandise $q_1, q_2, \cdots, q_s$ which he is engaged in exchanging. The value of this assumption might be questioned. The function $\phi(q_1, q_2, \cdots, q_s)$ is called

by M. Pareto the ophélimité of the individual considered. Under a monopoly, the prices are again mathematically determined but not to the greatest satisfaction of all engaged in the transaction, but to the advantage of the proprietors of the monopoly. If there is a maximum of "ophelimité," the cost of production is equal to services rendered in production.

Many other applications are given in the book. Among these should be mentioned Cournot's theory of exchange, the rôle of the theory of games of chance in statistics, which includes the questions of annuities and insurance.

Taken as a whole, the book is useful for the clearness of presentation as well as for the numerous applications to economic theory. While the reviewer would expect a treatise on statistics to contain more recognition of the recent work of Karl Pearson and those associated with him, the present book contains much valuable material for the student of mathematical statistics.

H. L. RIETZ.

Σώζειν τὰ Φαινόμενα. Essai sur la Notion de Théorie physique de Platon à Galilée. Par P. DUHEM. Paris, A. Hermann et Fils, 1908. 144 pp.

OFTEN, when fatigued with the perplexities of modern physics or the intricacies of modern mathematics, it is a pleasant change to take a dilettante interest in the science of the ancients, to draw an optimistic courage from the progress twenty centuries have made or a pessimistic cheer from the little that so long a time has won. Then a volume of Pliny or parts of Plutarch's works suggest themselves — in a translation, alas! despite or to spite eight years of Latin and six of Greek. There we can find a dissertation on flesh eating which reads like some of all too recent date or a disquisition on the moon and her inhabitants that seems quite modern Martian. The philosophers who live much by and with and for the Greeks have collected, collated, and translated the words of philosophic wisdom of these ancients. If such a collection should be made for science with some appropriate comments relative to our present point of view, a highly entertaining book could be printed. Perhaps Duhem will sometime get to this; his present work with its Greek title and French subtitle is merely an essay on the conception of physical theory from Plato to Galileo — and by physical theory is apparently meant only such as regards astronomy, the best developed of the Greek physical sciences.

Duhem points out that at the very outset in the days of Plato and Aristotle there were two distinct and largely contrary attitudes toward astronomical science. The first was the mathematical (or astronomical) point of view that the test of a hypothesis relative to the motion of the planets was merely whether or not that hypothesis enabled one to describe the facts — *σώζειν τὰ φαινόμενα*. The second was the metaphysical (physical) point of view which called upon a hypothesis to conform to the real nature of things — *κατὰ φύσιν* — whatever that may mean. Of course it is at once apparent to everybody that there probably always have been, surely are now, and very likely always will be these two points of view; that the scientist will incline to the first and the metaphysician to the second; that some will be content to make use of nature as best they can, while others will not be satisfied unless they make nature according to their own best ideas. Not only do the two points of view correspond to two different types of mind; they appeal to different moods of the same mind. A new theory seems naturally a convention and an old theory equally a reality. Osiander might well maintain in his preface to Copernicus's work that the Copernican view was merely a clever but unreal way to explain appearances. Yet it was not long before the adherents of the theory were willing to make martyrs of themselves for its reality and still to maintain: E pur si muove! It probably takes almost as much courage now-a-days to maintain that "the earth moves" means merely that "it is more convenient to assume that the earth moves."

Throughout the essay Duhem carefully traces the history of the conflict between the scientific and the metaphysical view. That the text is readable and entertaining may be taken for granted when it is written by the author of such a variety of well known works. It is interesting to note that during the fourteenth, fifteenth, and sixteenth centuries the masters at the Sorbonne set forth views on physical theory which were better and deeper than any heard up to the middle of the last century. The author's conclusion is also noteworthy, namely: En dépit de Kepler et de Galilée, nous croyons aujourd'hui, avec Osiander et Bellarmin, que les hypothèses de la Physique ne sont que des artifices mathématiques destinées à sauver les phénomènes; mais grace à Kepler et à Galilée, nous leur demandons de sauver à la fois tous les phénomènes de l' Univers inanimé. Perhaps there are still a number of unenlightened

physicists who can not take quite this view with respect to our more firmly fixed theories of physics; probably the majority of metaphysicians would not acquiesce. It may be that some day reality will consist merely of conventions, or it may be that the pendulum will again swing to the other side and make the conventions real. At any rate much will still be spoken and written on both sides of the question.

E. B. WILSON.

The Slide Rule. An Elementary Treatise. By J. J. CLARK. Technical Supply Company, Scranton, Pa., and New York. 62 pp.

ONE familiar with algebra and logarithms usually needs little instruction in the use of the slide rule. He needs only practice in reading results accurately and expeditiously. On the other hand a person ignorant of logarithms finds the mastery of the instrument a more difficult task. The author of this booklet directs his attention to the wants of the latter class. His aim is to give directions for the use of the slide rule, so simple and explicit that any pupil with a fair knowledge of arithmetic can understand them. In this laudable purpose the author has been eminently successful. The booklet is a model of clear exposition.

The author confines his attention to two slide rules, the Mannheim rule and the Rietz rule. The term "Mannheim rule" has become generic. The Mannheim type is now used more than any other for ordinary purposes, and is manufactured by many firms in different countries. The name Rietz is attached to a specific rule, manufactured by the firm of Albert Nestler in Lahr, Baden. The Rietz rule is one of the very numerous rules with the Mannheim arrangement of the lines A, B, C, D, to which one or more other lines are added (in this case the E line for cube root, etc.). Just why this Rietz rule should have been selected out of a very large number of similar domestic and foreign makes is not quite evident.

The author gives nothing on the history of the slide rule. It is perhaps just as well that no attempt should have been made in this line. Only very recently have I been able to settle the long-disputed question as to the inventor of the straight-edge slide rule.* Mr. Clark gives in his book just

* "History of the logarithmic slide rule," Engineering News Publ. Co., New York, 1909.

four words of biographical information, but in so doing he comes to grief. Mannheim was not "a German army officer." As a young man he was a French army officer, and for about half a century he was professor in the Polytechnic School in Paris.

FLORIAN CAJORI.

Annuaire du Bureau des Longitudes pour l'An 1910, Paris. Gauthier-Villars.

THIS handy little volume makes its annual appearance as usual in good time. So far as we know there is nothing approaching to it in completeness for giving the astronomical, physical, electrical, meteorological, magnetic, and chemical constants. And what is of even more importance, everything is easy to find and the matter is constantly kept up to date. For the construction of problems in which one wishes to insert numerical data, the hunting in various books can be avoided and much time saved by having the *Annuaire* at hand. Two consecutive volumes give everything likely to be needed.

The principal appendix this year is an article on earth tides and the elasticity of the globe by M. Ch. Lallemand. A full historical survey of our knowledge up to the present time forms one of its most useful features. M. B. Baillaud tells of the work done at the last international conference on the photographic chart of the sky, and M. G. Bigourdan appends a list of all the Notices which have appeared in the *Annuaire* since its foundation in 1796.

ERNEST W. BROWN.

NOTES ON THE INSTITUT DE FRANCE AND THE ANNUAL MEETING OF THE ACADÉMIE DES SCIENCES.

IT is not generally known in America that the Institut de France is made up of five distinct bodies, known as the Académie Française, the Académie des Sciences, the Académie des Inscriptions et Belles-Lettres, the Académie des Sciences Morales et Politiques and the Académie des Beaux-Arts. Each of these academies has 40 members, with the exception of the Académie des Sciences which — in addition to its two permanent secretaries, one in each of the departments, physics and

mathematics — has 66 members, six in each of the eleven sections: geometry, mechanics, astronomy, geography and navigation, general physics, chemistry, mineralogy, botany, agriculture, anatomy and zoology, medicine and surgery. Each academy appoints its own officers, elects its own members and each, with the exception of the Académie Française has, in addition to its regular members, "académiciens libres," "associés étrangers," and "correspondants." The term "forty immortals" is exclusively applied to the members of the Académie Française, which however is much less eminent in the matter of erudition than its less widely celebrated fellows. Its chief official function is to prepare a dictionary, for which it has no exceptional philological equipment. The other academies are extremely active and have great influence on the progress of learning. The weekly *Comptes Rendus* edited by the secrétaires perpétuels and published by the Académie des Sciences is known to all mathematicians.

The following are the regular academicians in the sections of geometry, mechanics, and astronomy:

Geometry: Appell, Humbert, Jordan, Painlevé, Picard, Poincaré.

Mechanics: Boussinesq, Deprez, Léauté, Levy, Sebert, Vieille.

Astronomy: Baillaud, Bigourdan, Deslandres, Hamy, Radau, Wolf.

Mathematics gets another representative through the fact that Darboux is a secrétaire perpétuel.

By way of closing this introductory note, it may be pointed out that the term Académie des Sciences is also used in quite a different sense, namely, as one section of the Académie de Paris or the Académie de Bordeaux or any other of the sixteen groups of educational institutions into which practically all schools (primary, secondary, and superior) in France are divided.

The annual meeting of the Académie des Sciences occurred on December 20, in the ancient circular assembly hall adorned with statues of Fenelon, Bossuet, Descartes, Sully, and Henri d'Orleans, Duc d'Aumale. An audience of several hundred had assembled when on the stroke of one o'clock the drums began to roll and the four gorgeously arrayed officers of the Académie (President Bouchard, Vice-President Émile Picard, Secrétaires perpétuels Darboux and Van Tieghem), followed by some fifty of the academicians, took their places. Throughout the two-hour programme, the officers remained seated behind

the table on the raised platform and occasionally refreshed themselves from the conspicuous carafe of water and bowl of sugar. With the words "La séance est ouverte," the president, who is a member of the Faculty of Medicine in the University of Paris, commenced to read his "allocution." Before proceeding to his main theme, which was a discussion of the decrease in birth rate of the French population, he announced that the only break in the ranks of the academy during the year was caused by the death of their illustrious "associé étranger," Simon Newcomb. Drawing freely from his "Reminiscences of an Astronomer," which he compared with the autobiographies of J. J. Rousseau and Franklin, M. Bouchard sketched Newcomb's early career down to the age of 23 when he was made Docteur ès Sciénces. He then concluded as follows: "I can not follow him in his glorious career as astronomer. His discoveries of the retrograde movement of Hyperion, his theories of the four inner planets suffice to place him in the first rank. He is considered as the follower of Leverrier . . . . We elected him our associé étranger in 1903." From the representative of one of the greatest scientific societies in the world, one might well expect greater accuracy of statement. Newcomb received the degree of *Bachelor* of Science from Harvard in 1858, and was elected one of the eight associés étrangers on June 17, 1895, in place of von Helmholtz. Franklin was the only American to be so honored previously, and Agassiz alone remains now. By a decree of December 1, 1909, the number of associés etrangers was increased from eight to twelve.

M. Van Tieghem now announced the award of medals and money prizes totalling close upon 235,000 francs. The awards as far as they were of interest to mathematicians are as follows: Prix Francoeur (3,000 fr.) to M. Émile Lemoine for his mathematical works.—Prix Bordin (3,000 fr.). The problem proposed for the prize was: "The absolute invariant which represents the number of double integrals of the second kind on an algebraic surface depends upon a relative invariant ρ, which plays an important part in the theory of integrals of total differentials of the third kind and in the theory of algebraic curves traced upon the surface. It is proposed to make an exhaustive study of this invariant and to determine in particular how its exact value can be found, at least for an extended category of surfaces." Competing memoirs were to be in the hands of the secretary before January 1, 1909, but the prize was awarded to

Professors Guiseppe Bagnera, of the University of Palermo, and Michele de Franchis, of the University of Catania, for their memoir entitled "Le nombre ρ de M. Picard pour les surfaces hyperelliptiques et pour les surfaces irregulières de genre zéro," which did not arrive until January 15. It was stated that this case must not be thought to create a precedent, the circumstances being as follows: "The memoir was sent from Palermo on the evening of December 27, 1908, and it was in the following night that the awful catastrophe overtook Messina. The postal car containing the manuscript was in the station of that city at the moment of the earthquake, and nearly three weeks elapsed before it was despatched further." — Prix Montyon (700 fr.) to M. Lecornu, chief engineer of mines, professor in the École Polytechnique, for his work entitled Dynamique appliquée. — Prix Poncelet (2,000 fr.) to M. de Sparre for his studies relating to gunnery and his works on mechanics. — Prix Vaillant (4,000 fr.) for the subject: "To perfect in an important point the application of the principles of the dynamics of the fluids to the theory of the helix" was not awarded. — Prix Boileau (1,300 fr.) to M. Boulanger, assistant professor of mechanics, University of Lille, for his work entitled Hydraulique générale. — Prix Lalande (1,000 fr.) to M. Borrelly, assistant astronomer, observatory of Marseilles, for his discoveries of small planets and comets. — Prix Valz (460 fr.) to M. de la Baume-Pluvinel for his astronomical works. — Prix G. de Pontécoulant (700 fr.) to Professor E. W. Brown, of Yale University, for his researches relative to the theory of the moon. — Prix Binoux (2,000 fr.) to M. Pierre Duhem, correspondent of the academy, for his works relative to the history of the sciences. — Fondation Leconte. 1,200 fr. to M. Ritz for his works in mathematical physics and mechanics; 2,000 fr. to M. Lebeuf, director of the Besançon observatory for his chronometric and astronomical works and in particular for his part in the publication of the works of Laplace. — Prix Laplace (Laplace's works) to M. A.V.É. Vaucheret, leader of the graduating class at the École Polytechnique. — From the Bonaparte Fund 2,000 francs were granted to M. Estanave, D. Sc. Math., secretary of the Société mathématique de France, and author of several bibliographical and mathematical works, for continuing his researches "sur la projection stéréoscopique à vision directe, sur la stéréo-radiographie et sur l'autostéréoscopie;" and 2,000 francs to M. Mathias, professor

of physics at the University of Toulouse, to permit him to pursue, in the Leyden laboratory, with its eminent director, M. Kammerlingh Onnes, researches on the rectilinear diameter of liquids and on the law of state at very low temperatures.

The following prizes to be awarded in 1910 and 1911 were announced:

1910. Grand Prix (3,000 fr.). "It is known how to find all systems of two meromorphic functions of one variable, connected by an algebraic relation. An analogous problem is proposed for three uniform functions of two complex variables having for finite regions the character of rational functions and being connected by an algebraic relation. In case a complete solution is not obtained, the Academy demands at least some illustrative examples which lead to classes of new transcendental functions." — Prix Poncelet (2,000 fr.) for a work in pure mathematics. — Prix Francoeur (1,000 fr.) for a work of merit in pure or applied mathematics. — Prix Fourneyron (1,000 fr.) for an experimental and theoretical study, in hydraulics, of the effects of the shocks (coups de bélier) given to elastic tubes. — Prix Janssen, biennial gold medal for the discoverer or the work making important advancement in physical astronomy.

1911. Prix Bordin (3,000 fr.). "To perfect in an important point the theory of triply orthogonal surfaces." The Academy desires to add to the list of systems already known, and it attaches a particular prize to the discovery of the triply algebraic systems which are most simple. — Prix Vaillant (4,000 fr.). To perfect in some important point the study of the movement of an ellipsoid in an infinite viscous liquid. — Prix Damoisea To perfect Leverrier's Tables of Jupiter.

Darboux now read his long but interesting "historical notice" of General Jean Baptiste Marie Charles Meusnier (1754–1793), member of the Academy of Sciences. His scientific and military career were traced in detail and the celebrated theorem concerning the curvature of oblique sections of surfaces, with which his name is always associated, was assigned to the year 1776. The s in Meusnier's name is frequently omitted by English writers. May it now be correctly preserved.

With the President's announcement "La séance est levée" the meeting came to an informal ending.

R. C. ARCHIBALD.

PARIS, *December* 24, 1909.

NOTES.

THE opening (January) number of volume 32 of the *American Journal of Mathematics* contains the following papers: "The complementary theorem," by J. C. FIELDS; "The twelve surfaces of Darboux and the transformation of Moutard," by L. P. EISENHART; "On the problem of the spherical representation and the characteristic equations of certain classes of surfaces," by A. E. YOUNG; "The general circulation of the atmosphere," by F. R. SHARPE; "Generalizations of the tetrahedral and octahedral groups," by G. A. MILLER; "The theory of degenerate curves and surfaces," by O. E. GLENN.

THE January number (volume 11, number 2) of the *Annals of Mathematics* contains the following papers: "Necessary and sufficient conditions that ordinary differential equations shall admit a conformal group," by L. I. HEWES; "The three-space projective geometry (3, 2) and its group," by G. M. CONWELL; "The geodesic lines on the helicoid," by S. E. RASOR: "Cubic congruences with three real roots," by E. B. ESCOTT.

AT the meeting of the London mathematical society held on January 13 the following papers were read: By H. BATEMAN, "The transformations of coordinates which can be used to transform one physical problem into another"; by W. H. YOUNG, "On homogeneous oscillation"; by W. H. and Mrs. G. C. YOUNG, "On the determination of a semicontinuous function for a countable set of values"; by G. H. HARDY, "Note on a former paper on the theory of divergent series"; by H. F. BAKER, "On the expression of a certain function by means of a series of polynomials," and "On the theory of the cubic surface"; by G. N. WATSON, "The harmonic functions associated with the parabolic cylinder."

THE British mathematical association and the Association of public school science masters have appointed a joint committee to consider the possibility of correlating the teaching of mathematics and the sciences. The committee made its report on January 12, but the associations wish to consider it further before the contents are made public.

THE Euler commission announces that the editorial committee will consist of Professor F. RUDIO, of the technical school at Zürich, chairman, and Professors A. KRAZER and P. STAECKEL, of the technical school at Karlsruhe. The first volume of the works will be devoted to algebra and will be edited by Professor H. WEBER, of the University of Strassburg. As now planned, the entire series will consist of 43 quarto volumes of about 500 pages each. The 18 volumes in pure mathematics are distributed as follows: arithmetic and algebra, 5 volumes; analysis, 11 volumes; geometry, 2 volumes. Mechanics occupies 11 volumes.

THE list of German doctorates in mathematics for the years 1906–1908 which appeared in the February number of the BULLETIN was compiled by Mr. DUNHAM JACKSON from the Verzeichnis der an den deutschen Universitäten erschienenen Doktor-Dissertationen und Habilitationsschriften. A continuation of the list will appear in an early number of the BULLETIN.

THE Central Committee of the International Commission on the teaching of mathematics, consisting of Professors Klein, Greenhill, and Fehr, held a meeting at Basel on December 28, 1909. Reports received on the present state of the work in the eighteen participating countries showed that the investigation was proceeding with much energy and that it had enlisted the services of a large number of prominent teachers and mathematicians. The Central Committee will hold its next meeting at Brussels about the middle of August, 1910. It is hoped that the international exhibition to be held there at that time will bring together a considerable number of official delegates of the commission.

THE annual meeting of the council of the American Federation of teachers of the mathematical and natural sciences was held on Monday, December 27, 1909, at the Massachusetts Institute of Technology, Boston, Mass. Twenty-two representatives of eight associations were present. The report of the executive committee showed that six associations had joined the Federation during the preceding year. The total number of paid up members in the associations belonging to the Federation is now 2040. Reports were presented from the local associations showing activity and progressive work in all. The committee on a syllabus in geometry reported that work was well

under way. The committee has been divided into three subcommittees, one on logical considerations, one on lists of basal theorems, and one on exercises and applications. It expects to have its work completed during the present year.

The committee on college entrance requirements had gathered a large amount of information which showed a great variation in the requirements of the different colleges, making it impossible for any school to meet them all. The committee recommended that the Federation take up this matter with the College entrance examination board and see what can be done toward bringing about uniformity. The report was accepted and the committee continued.

It was urged that reports of local meetings be published in such periodicals as would best serve the associations concerned.

A committee was appointed to cooperate with the College entrance examination board to determine the best forms of logarithmic tables to be used at examinations.

The question of the publication of a journal for mathematics alone was discussed at some length, and it was voted that a committee be appointed to consider this question and report at the next meeting.

Informal reports of progress were presented by members of the International commission on the teaching of mathematics.

The following officers were elected for the present year: President, Professor C. R. MANN, of the University of Chicago; secretary and treasurer, Professor E. R. SMITH, of the Brooklyn Polytechnic Institute; and three other members of the executive committee. This committee will determine the time and place of the next meeting.

STATISTICS presented by Professor A. SCHOENFLIES in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* show that the number of students of mathematics (excluding foreigners) in the Prussian universities has increased from 1,440 in 1907 to 1,730 in 1910, the latter number exceeding all previous records. At present the supply of available graduates meets about one-half the demand for teachers, but warning is given that less favorable conditions are likely to result from a continued increase in the number of students.

PROFESSOR R. DEDEKIND, of the technical school at Braunschweig, has received the honorary degree of doctor of mathematics from the technical school at Zürich.

PROFESSOR E. SCHMIDT, of the University of Zurich, has been appointed to a full professorship of mathematics at the University of Erlangen.

DR. I. SCHUR, of the University of Berlin, has been promoted to an associate professorship of mathematics.

MISS E. GREENE has been appointed mathematical tutor at Bedford College, of the University of London.

MR. A. J. KENNY has been appointed assistant lecturer in mathematics at the University of Birmingham.

MR. J. H. SLEEMAN has been appointed lecturer in mathematics at the University of Sheffield.

MR. M. J. CONRAN has been appointed lecturer in mathematical physics at the University College of Cork.

PROFESSOR W. W. CAMPBELL, of the Lick Observatory, delivered at Yale University, on the Silliman foundation, in the week beginning January 24, a course of eight lectures on "Stellar motions."

DURING the week February 4–10, Professor CARL RUNGE, of the University of Göttingen, delivered a course of five lectures on graphical methods at the University of Michigan.

PROFESSOR C. J. KEYSER, of Columbia University, is spending a half-year abroad, on leave of absence.

DR. E. G. BILL, of Yale University, has received leave of absence for the coming academic year, which he will spend in the study of geometry at the University of Turin.

PROFESSOR L. A. WAIT, head of the department of mathematics at Cornell University, will retire from active service at the close of the present academic year.

EDWARD A. BOWSER, professor emeritus of mathematics and engineering at Rutgers College, died at Honolulu on February 29 at the age of sixty-five years.

PROFESSOR H. B. NEWSON, of the University of Kansas, died on February 18, at the age of fifty years. Professor Newson graduated from Ohio Wesleyan University in 1883. Later he studied at Johns Hopkins, Heidelberg, and Leipzig. He became associate professor of mathematics at the University

of Kansas in 1890, and was promoted to a full professorship in 1905. He had been a member of the American Mathematical Society since 1895.

PROFESSOR J. E. WRIGHT, of Bryn Mawr College, died on February 20 after an illness of several months. Professor Wright was senior wrangler of the University of Cambridge in 1900, first in the second part of the mathematical tripos in 1901, Smith's prizeman in 1902, and fellow of Trinity College since 1903. He was called to Bryn Mawr in 1903. His treatise on Invariants of Quadratic Forms was published by the Cambridge University Press in 1908.

NEW PUBLICATIONS.

(In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor, PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.)*

I. HIGHER MATHEMATICS.

BERWALD (L.). Krümmungseigenschaften der Brennflächen eines geradlinigen Strahlensystems und der in ihm enthaltenen Regelflächen. (Diss.) München, 1909. 8vo. 67 pp.

BÖHM (F.). Parabolische Metrik im hyperbolischen Raum. (Diss.) München, 1908. 8vo. 62 pp.

BOJKO (J.). Neue Tafel der Viertelquadrate aller natürlichen Zahlen von 1–20000 zur Bildung aller möglichen Produkte im Bereiche $1 \times 1 —$ 10000×10000. Zürich, Speidel, 1909. 8vo. 20 pp. M. 1.50

DEGENHART (H.). Ueber einige zu zwei ternären quadratischen Formen in Beziehung stehende Konnexe. (Diss.) München, 1909. 8vo. 55 pp.

DENJOY (A.). Sur les produits canoniques d'ordre infini. (Thèse.) Paris, Gauthier-Villars, 1909. 4to. 141 pp.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome I (volume 1): Arithmétique. Fascicule 4: Schoenflies et Baire, Théorie des ensembles; Burkhardt et Vogt, Sur les groupes finis discontinus. Paris, Gauthier-Villars; Leipzig, Teubner, 1909. 8vo. Pp. 489–616. Fr. 5.00

—— Tome I (vol. 4): Calcul des probabilités; théorie des erreurs; applications diverses. Fascicule 4: Mehmke et d'Ocagne, Calculs numériques; Bortkiewicz et Oltramare, Statistique. Paris, Gauthier-Villars; Leipzig, Teubner, 1909. 8vo. Pp. 321–480. Fr. 6.00

GRAND (J.). Anwendung der Lindstedtschen Methode auf die Integration der Differentialgleichung für hin- und hergehende Bewegungen eines zwangläufigen Mechanismus. (Diss.) Zürich, 1908. 8vo. 52 pp.

HAAR (A.). Zur Theorie der orthogonalen Funktionensysteme. (Diss.) Göttingen, 1909. 8vo. 48 pp.

HAWLITSCHEK (K.). Kreismessung. (Progr.) Prag, 1909. 8vo. 20 pp.

HOWLAND (L. A.). Anwendung binärer Invarianten zur Bestimmung der Wendetangenten einer Kurve dritter Ordnung. (Diss.) München, 1908. 8vo. 93 pp.

JURISCH (K. W.). Beweis des Fermatschen Satzes. Berlin, Heymann, 1909. 8vo. 12 pp. M. 1.60

KOCH (H.). Ueber die praktische Anwendung der Runge-Kuttaschen Methode zur numerischen Integration von Differentialgleichungen. (Diss.) Göttingen, 1909. 8vo. 37 pp.

MEYER (C.). Zur Theorie des logarithmischen Potentials. (Diss.) Leipzig, 1909. 8vo. 67 pp.

NADLER (C.). Ueber den Zusammenhang der Raumkurve vierter Ordnung erster Spezies mit ihrem Polartetraeder. (Diss.) Rostock, 1909. 8vo. 38 pp.

PLANCHEREL (M.). Sur les congruences (mod. 2^m) relatives au nombre des classes des formes quadratiques binaires aux coefficients entiers et à discriminant négatif. (Diss.) Freiburg, 1909. 8vo. 93 pp.

POINCARÉ (H.). Vorträge über ausgewählte Gegenstände aus der reinen Mathematik und mathematischen Physik, auf Einladung der Wolfskehl-Kommission der königlichen Gesellschaft der Wissenschaften gehalten zu Göttingen vom 22. bis 28. IV. 1909. Leipzig, Teubner, 1910. 8vo. 4 + 60 pp. M. 2.40

RENNER (J.). Konforme Abbildungen. (Progr.) Gray, 1909. 8vo. 31 pp.

SCHMID (A.). Anwendung der Cauchy-Lipschitzschen Methode auf lineare partielle Differentialgleichungen. (Diss.) München, 1908. 8vo. 30 pp.

SPEISER (A.). Die Theorie der binären quadratischen Formen mit Koeffizienten und Unbestimmten in einem beliebigen Zahlkörper. (Diss.) Göttingen, 1909. 8vo. 33 pp.

THIELE (T. N.). Interpolationsrechnung. Leipzig, 1909. 4to. 187 pp. M. 10.00

II. ELEMENTARY MATHEMATICS.

BARBISCH (H.). See JAHNE (J.).

BAUER (W.) und HANZLEDEN (E. VON). Lehrbuch der Mathematik, zum Gebrauche an höheren Mädchenschulen bearbeitet. 1ter Band. Planimetrie und Arithmetik: Pensum von Klasse IV und Klasse III. Braunschweig, Vieweg, 1910. 8vo. 11 + 147 pp. M. 2.40

BESSON (L.). Algèbre, troisième année (candidats aux arts et métiers). Macon, Perraux, 1909. 8vo. 120 pp.

BROOKS (E.). The normal standard algebra; designed for public and private schools, normal schools, academies, etc. Philadelphia, Lower, 1909. 12vo. 8 + 431 pp. Cloth. $1.22

DREYFUS (L.). See FORT (L.).

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